Classes, The Analytical Hierarchy and Shoenfield's absoluteness theorem

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Abstract

In this note we will introduce some standard notation for the definability-complexity of classes, and we will prove Shoenfield's absoluteness theorem.

This note is a suplementary text to the curriculum of MAT4640 Axiomatic Set Theory. The main textbook is Kunen [1]. We will assume that the reader is familiar with relevant chapters from Kunen [1].

The main result in this note is the so called *Shoenfield absoluteness theorem* stating that all Σ^1_2-statements are absolute for Gödel's L. We assume that the reader knows about L, but the meaning of Σ^1_2 will be explained.
It is sometimes useful to classify classes and sets by the form of the simplest definitions, and there is a standard notation for families of classes or sets. One aim in this note is to introduce the reader to this notation and to the basic properties of definability classes of sets or classes.

We will use separate enumerations for lemmas, theorems, definitions and exercises in this note.

1 The class hierarchy

1.1 Syntax classes

First we will consider the class Δ_0 known from Kunen [1]. We will take the liberty, though, to extend the set of Δ_0-formulas. We will use the language of ZF extended with the connective $\vee$, the quantifier $\forall$ and the bounded quantifiers $\forall x \in y$ and $\exists x in y$. We do this for the sake of convenience, and we know that all statements in this extended language can be rephrased in the genuine minimalistic language of ZF. The use of these extra quantifiers makes our definition of syntax classes simpler.
We will not include $\to$ and $\leftrightarrow$ in our language, though, but use the symbols for expressing abbreviations as before.

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Definition 1 A Δ_0-formula is a formula in this extended language where we do not use the unbounded quantifiers $\exists x$ and $\forall x$.
A Δ_0-class is a class definable by a Δ_0-formula.

Exercise 1 Show that every Δ_0-frmula will contain at least one free variable.

Remark 1 Recall that a class strictly spoken is a defining formula, where we may use parameters. Since we consider two classes to be equal if the defining formulas are equivalent, we may think of a class as the collection objects making the defining formula true. In this note we will only claim that two classes are equal if they are provably equivalent, either by first order logic alone, or by he axioms of ZF. In the latter case, it may be of interest to find out how much of ZF we need in order to prove the equivalence.

We will allow multi-dimentional classes. This is of course nothing else than classes where all elements are ordered sequences, in the sense of set theory, of a fixed length. It simplifies the exposition, tough, to consider classes defined by formulas with more than one free variable.

Definition 2 We define the hierarchy of formulas as follows:

1. A formula Φ is Π_0 or Σ_0 if it is Δ_0.
2. Φ is a Σ_{n+1}-formula if it is of the form $\exists x_i \Psi$ where Ψ is a Π_n-formula.
3. Φ is a Π_{n+1}-formula if it is of the form $\forall x_i \Psi$ where Ψ is a Σ_n-formula.

Definition 3 A class is Π_n if the defining formula is equivalent to a Π_n-formula.
A class is Σ_n if the defining formula is equivalent to a Σ_n-formula.
A class is Δ_n if it is both Π_n and Σ_n.

Remark 2 Unless we have a very Platonic view on the universe of sets, we must be careful here. In order to see that a class is Π_n, Σ_n or Δ_n we normally have to prove the equivalence from a fragment of ZF or from some extra axioms. In these notes we will consider equivalences provable in ZF or in $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$.

In the sequel, we will mainly be interested in these families when $n = 0$ or $n = 1$, but we take the opportunity to prove some basic, and useful, properties of these families of classes in general.

1.2 Closure properties

It is well known from first order logic that any formula can be rewritten to an equivalent formula on prenex normal form, i. e. a formula where all quantifiers appears in front in a *prefix* and the propositional part appears to the right in a *matrix*. Since our underlying logic is first order logic, we may assume without loss of generality that all our defining formulas are on prenex normal form.
This does not mean that they are Π_n or Σ_n for some n, since this requires that

all unbounded quantifiers appear to the left of all bounded quantifiers, and that the unbounded quantifiers alternate in form. Consider the class

$$X = \{x \mid \exists y \exists z(y \in x \wedge x \in z\}$$

The defining formula is not in our hierarchy of syntactical forms, but the class is indeed Δ_0, provably in a very small fragment of ZF. We will now prove a lemma that gives us a way to transcribe any formula to either a Π_n formula or to a Σ_n formula for some n. This gives us an easy way to place any class in one of our syntax classes. Of course, we may sometimes use our mathematical ability to improve the results obtained by this simple strategy.

Lemma 1 *The following transitions lead to provably equivalent formulas:*

1. *Replace* $\exists x \exists y$ *with* $\exists z \exists x \in z \exists y \in z$.
2. *Replace* $\forall x \in z \exists y$ *with* $\exists u \forall x \in y \exists y \in u$.
3. *Replace* $\forall x \forall y$ *with* $\forall z \forall x \in z \forall y \in z$.
4. *Replace* $\exists x \in z \forall y$ *with* $\forall u \exists x \in z \forall y \in u$.

Proof
3. follows from 1. and 4. follows from 2. by propositional calculus.
1. is a consequence of the pairing axiom.
2. follows from the replacement axiom and the **WF**-axiom. We need the **WF**-axiom to rephrase the replacement axiom scheme to the scheme

$$\forall x \in u \exists y \Phi(x, y) \to \exists v \forall x \in u \exists y \in v \Phi(x, y).$$

We leave the proof of the correctness of the rephrasing as an exercise for the reader. With this rephrased version, 2. is trivial.

We do of course have similar transition rules for combined quantifiers of the form $\exists y \in z \exists x$ and $\forall y \in z \forall z$, leaving the precise formulation as an exercise for the reader.
This gives us all we need in orderto prove the following

Theorem 1 *If* $n > 0$*, the family of* Σ_n *classes of a fixed arity will be closed under*

- *finite intersections*
- *finite unions*
- *bounded quantifiers*
- *unbounded existential quantifiers.*

For the family of Π_n*-classes we replace the last item with "unbounded universal quantifiers".*

Definition 4 A *class function* F is a class $F(\vec{x}) = y$ that satisfies

$$\forall \vec{x} \exists ! y (F(\vec{x}) = y).$$

We say that F is $\Sigma_1, \Pi_1, \ldots$ etc. if the class $F(\vec{x}) = y$ is $\Sigma_1, \Pi_1, \ldots$ resp.

Lemma 2

1. *If F is a Σ_n class function then F is a Δ_n class function.*
2. *Composition of Δ_1-functions uields a Δ_1-function.*
3. *If F is Δ_1 and G is defined from F by transfinite recursion, then G is Δ_1.*

Proof

1. This is trivial if $n = 0$, so let $n > 0$ and let $F(\vec{x}) = y$ when $\exists z \Phi(\vec{x}, y, z)$ where Φ is in Π_{n-1}. Since F is a class function we will have

   $$F(\vec{x}) = y \Leftrightarrow \forall z \forall u (\Phi(x, u, z) \Rightarrow u = y).$$

   Using prenex operations and the transitions from Lemma 1 we get an equivalent formula on Π_n-form.
2. Since composition of Σ_1-relations will be Σ_1, we may use 1. of this lemma.
3. When we showed that G is a class function when defined by transfinite recursion from a class function F we used one unbounded existential quantifier, demanding that there is a partial g, satisfying the recursion on its domain, such that $g(\vec{x}) = y$.
   When F is Δ_1, the statement that g is a partial solution is also Δ_1, so G is Σ_1. By 1. of this lemma, G is Δ_1.

One important special case of this is that we may use constants with a Δ_1 definition in a formula without ruining that the defined class is Δ_1.

Exercise 2 Let

$$OP(x, y)\{\{x, y\}, \{x\}\}.$$

Let $R \subseteq \mathsf{V} \times \mathsf{V}$ be a class. Let

$$S = \{OP(x, y) \mid (x, y) \in R\}.$$

Let $n > 0$ and show that R is Σ_n if and only if S is Σ_n.
Prove the same statement for Π_n with $n > 0$.
Do we need that $n > 0$ in any of these directions?

A closer inspection of the construction of L will tell us that we have proved what we need for the following

Corollary 1 *The function $\alpha \mapsto \mathsf{L}(\alpha)$ is Δ_1 and L is Σ_1.*

Proof
$\mathsf{L}(\alpha)$, viewed as a function, is constructed by iterated use of transfinite recursion, the constant ω, the pairing function and other similarly simple operators. This gives us that the function is Δ_1. The class ON is also Δ_1 and

$$x \in \mathsf{L} \Leftrightarrow \exists y \exists z (y \in \mathsf{ON} \wedge x \in z \wedge z = \mathsf{L}(y)).$$

Thus L is Σ_1.

1.3 Universal classes

We have seen that it is hard to get out of Δ_1, and one might suspect that with some cleverness we might prove that all classes are Δ_1. We will now show that this is not the case, that on the contrary, there will be new classes at any level of our hierarchy.

Exercise 3 Show that all Σ_n classes and all Π_n-classes are Δ_{n+1}-classes. Show that the complement of a Σ_n-class is a Π_n-class and vice versa.

We will now show that if $n > 0$ then neither of the families Σ_n nor Π_n are closed under complement. We will do so by constructing so called universal classes in each family, and then, by a diagonal argument, find an element not in the family where the complement is in the family.
We need a Gödel enumeration of the Δ_0-formulas as a tool. This definition cannot be formalized in ZF since we make direct reference to the syntax of ZF. The purpose will be to use the arithmetisation to formally define a truth predicate for Δ_0-formulas.

Definition 5 For each $n, k \in \mathbb{N}$ with $k \geq 1$ we define a Δ_0-formula $\phi_{n,k}(x_1, \ldots, x_k)$ as follows:

1. If $n = 2^0 3^i 5^j$ and $1 \leq i, j \leq k$, let $\phi_{n,k}$ be $x_i = x_j$.
2. If $n = 2^1 3^i 5^j$ and $1 \leq i, j \leq k$, let $\phi_{n,k}$ be $x_i \in x_j$.
3. If $n = 2^2 3^i 5^j$, let $\phi_{n,k}$ be $\phi_{i,k} \vee \phi_{j,k}$.
4. If $n = 2^3 3^i 5^j$, let $\phi_{n,k}$ be $\phi_{i,k} \wedge \phi_{j,k}$.
5. If $n = 2^4 3^i$, let $\phi_{n,k}$ be $\neg\phi_{i,k}$.
6. If $n = 2^5 3^l 5^j$ and $i \leq k$, let $\phi_{n,k}$ be $(\exists x_{k+1} \in x_i)\phi_{j,k+1}$.
7. If $n = 2^6 3^i 5^j$ and $i \leq k$, let $\phi_{n,k}$ be $(\forall x_{k+1} \in x_i)\phi_{j,k+1}$.
8. Otherwise, let $\phi_{n,k}$ be $x_1 = x_1$.

If $a_1, \ldots, a_k$ are sets, we let $\langle a_1, \ldots, a_k \rangle$ be a standard coding of the ordered sequence $(a_1, \ldots, a_k)$ as a set. It does not matter if we use iteration of the ordered pair construction or functions $f : k \to \mathsf{V}$, only the fact that the class of coded ordered sequences is Δ_1 and that the functions giving the length of the sequence and the individual a_i are Δ_1.

Definition 6 Let $Pred(n, k, \langle a_1, \ldots, a_k \rangle)$ if and only if $\phi_{n,k}(a_1, \ldots, a_k)$.

Lemma 3 *$Pred$ is Δ_1-definable in ZF.*

Proof
$Pred$ is definable by recursion over ω using a Δ_1 function.

We then obtain

Theorem 2 *Let $k \geq 0$.*
Then there is a Σ_1-class Φ_k of arity $k+1$ such that for all Σ_1-classes Ψ of arity k there is an $n \in \omega$ such that for all sequences $\vec{x}$ of length k

$$\Psi(\vec{x}) \Leftrightarrow \Phi_k(n, \vec{x}).$$

Proof
Let

$$\Phi_k(x, \vec{x}) \Leftrightarrow (x \in \omega \wedge \exists y Pred(x, k+1, \langle y.\vec{x} \rangle)).$$

This clearly does the job

We say that Φ_k is *universal* for the family of Σ_1-classes of arity k. The negation rewritten to a prenex normal form will then be universal for all Π_n-classes.

Corollary 2 *Let Γ be the family Π_n or Σ_n for some $n \geq 1$.*
Then, for each $k \geq 0$ there is an element Φ_k in Γ of arity $k+1$ that is universal for all $\Psi \in \Gamma$ of arity k.

Proof
Let $\vec{Q}\vec{y}$ be the common quantifier prefix used for all classes in Γ. Let

$$\Phi_k(x, \vec{x}) = (x \in \omega \wedge \vec{Q}\vec{y} Pred(x, \langle \vec{x}, \vec{y} \rangle))$$

where we use the Σ_1-form or Π_1-form of $Pred$ that corresponds to the final quantifier in $\vec{Q}$.
This does the trick.

Corollary 3 *For each n there is a Σ_n-class that is not Π_n.*

Proof
Let $\Phi(x, y)$ be universal for all Σ_1-classes of arity 1. Let $\Psi(x) \Leftrightarrow x \in \omega \wedge \neg\Phi(x, x)$.
Then Ψ is Π_n, but distinct from all Σ_n classes.

Note that Ψ actually defines a set, a subset of ω. It is possible to find a Π_n-class that is not Σ_n definable from *any set*. We will not need this in this note, and leave the verification as a challenging exercise for the reader.

1.4 Absoluteness over HC

Recall that HC, the *hereditarily countable sets*, is the set of sets where the transitive closure is countable. Recall the definition of *absolute formulas* from Kunen [1].

Theorem 3 *All Σ_1-formulas and Π_1-formulas are absolute for* HC.

Proof
It suffices to prove this for Σ_1, since being absolute is closed under negation.
Let $\Phi(\vec{x}) = \exists x \Psi(x, \vec{x})$ where Ψ is Δ_0.
Let $\vec{a}$ be elements in HC. Since all Δ_0-formulas are absolute for all transitive

sets, $\Phi(\vec{a})$ will hold in V if $\Phi(\vec{a})$ holds in HC.
Assume that $\Phi(\vec{a})$ holds in V, and let b be a set such that $\Psi(b, \vec{a})$.
Let M be a transitive set such that $b \in M$ and $\langle \vec{a} \rangle \in M$.
By Löwenheim-Skolem there is a countable $M_0 \subseteq M$ that is elementary equivalent to M such that the transitive closure of each a_i from $\vec{a}$ are subsets of M_0.
Let $m : M_0 \to \mathsf{HC}$ be the Mostowski collapse of M_0.
Then $\Psi(m(b), \vec{a})$ will hold, and consequently, $\Psi(\vec{a})$ will hold in HC.

Recall that the power set axiom is the only ZF-axiom that fails for HC. We did not use the power set axiom in proving the closure properties of the classes Σ_n and Π_n. Thus all these results hold when relativized to HC. This is only of importance to us for $n = 1$.

2 The analytical hierarchy

In this section, we will discuss definability over the two sorted structure consisting of $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$. We will not spend space and time being overprecise about the language we use, as we in any case will assume that all our definitions are formalized in the language of ZF. Nevertheless, we will assume that we can express standard arithmetical functions and relations, that we have quantifiers over $\mathbb{N}$ and over $\mathbb{N}^{\mathbb{N}}$ and that we can express function application $f(x)$ when f varies over $\mathbb{N}^{\mathbb{N}}$ and x varies over $\mathbb{N}$.

Definition 7

1. A Δ^1_0-formula is a formula where all quantifiers are over $\mathbb{N}$.
2. We identify Π^1_0, Σ^1_0 and Δ^1_0, and call these formulas *arithmetical.*
3. A Σ^1_{k+1}-formula is a formula of the form $\exists f \in \mathbb{N}^{\mathbb{N}}\Phi$ where Φ is Π^1_k.
4. A Π^1_{k+1}-formla is ma formula of the form $\forall f \in \mathbb{N}^{\mathbb{N}}\Phi$ where Φ is Σ^1_k.
5. A subset of $\mathbb{N}^{\mathbb{N}}$ is called Σ^1_k resp. Π^1_k if it is definable by a Σ^1_k- resp, a Π^1_k-formula.
6. A subset $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Δ^1_k if A is both Σ^1_k and Π^1_k.

Remark 3 All these classes have a *relativized* variant, where we allow parameters from $\mathbb{N}^{\mathbb{N}}$ in the definitions.
A classical theorem due to Souslin (191) is that the relativized Δ^1_1-sets are exactly the Borel subsets of $\mathbb{N}^{\mathbb{N}}$. This is the starting point of desceiptive set theory, an interesting path that we will not follow.

We will now discuss the closure properties of the classes Π^1_k and Σ^1_k seen as classes of sets. We will systematically use n, m, etc. for elements of $\mathbb{N} = \omega$ and we will use f, g, etc. for elements of $\mathbb{N}^{\mathbb{N}}$.

Definition 8

1. Let $\langle n, m \rangle = \frac{1}{2}((n+m)^2 + 3n + m)$. If $k = \langle n, m \rangle$ we let $\pi_0(k) = n$ and $\pi_1(k) = m$.

2. Let $\langle f, g\rangle(n) = \langle f(n), g(n)\rangle$. If $h = \langle f, g\rangle$, we let $\pi_0(h) = f$ and $\pi_1(h) = g$.

3. If $\{f_i\}_{i\in\mathbb{N}}$ is a sequence of functions, let $\langle f_i\rangle_{i\in\mathbb{N}}(\langle j, m\rangle) = f_j(m)$. If $f = \langle f_i\rangle_{i\in\mathbb{N}}$ we let $(f)_n = f_n$. Note that $(f)_n$ is always defined.

4. If $n \in \mathbb{N}$ and $f \in \mathbb{N}^{\mathbb{N}}$ we let $\langle n, f\rangle(0) = n$ and $\langle n, f\rangle(m+1) = f(m)$, We use π_0 and π_1 here as well, assuming that no confusion arises.

These operators will be definable bijections between

- $\mathbb{N}^2$ and $\mathbb{N}$.
- $(\mathbb{N}^{\mathbb{N}})^2$ and $\mathbb{N}^{\mathbb{N}}$.
- $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.
- $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

We leave the verification of this for the reader. Indeed, any finite product of $\mathbb{N}$, $\mathbb{N}^{\mathbb{N}}$ and $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ will be definably equivalent to $\mathbb{N}^{\mathbb{N}}$ as long as the cardinality is uncountable. Observe that all versions of the projections π_0 and π_1 will be total.

Theorem 4 *The class of Σ^1_k sets for $k > 0$ will be closed under finite intersections finite unions, quantification over $\mathbb{N}$ and existential quantification over $\mathbb{N}^{\mathbb{N}}$. The class of Π^1_k-sets is closed under finite intersections and unions, quantification over $\mathbb{N}$ and universal quantification over $\mathbb{N}^{\mathbb{N}}$.*

Proof
We leave the details of the proof for the reader. The crucial steps are the transcriptions where

- $\exists f \exists g \cdots f \cdots g \cdots$ is replaced by $\exists h \cdots \pi_o(h) \cdots \pi_1(h) \cdots$
- $\forall f \forall g \cdots f \cdots g \cdots$ is replaced by $\forall h \cdots \pi_o(h) \cdots \pi_1(h) \cdots$
- $\forall n \exists f \cdots n \cdots f \cdots$ is replaced by $\exists f \forall n \cdots n \cdots (f)_n \cdots$
- $\exists n \forall f \cdots n \cdots f \cdots$ is replaced by $\forall f \exists n \cdots n \cdots (f)_n \cdots$

By definition, a Π^1_k-set or a Σ^1_k set can be defined by a formula on prenex normal form, where all function quantifiers will appear before all number quantifiers. In the sequel, we will make use of the fact that we will never need more than one number quantifier, and that one will be of the opposite kind as the innermost function quantifier. This claim is justified by the following

Lemma 4 *The formulas $\forall n \exists m \Phi(n, m)$ and $\exists g \forall n \Phi(n, g(n))$ are equivalent for all Φ.*

The proof is easy, and is left for the reader.

As a consequence, we get the *normal form theorem* for Π^1_1 sets:

Theorem 5 *All Π^1_1-sets are definable by a formula of the form*

$$\Phi(\vec{x}) = \forall f \exists n R(f, n, \vec{x})$$

where R is definable from computable functions using bounded quantifiers $\exists i < k$ and $\forall i < k$ only together with propositional connectives.

We will give an application in Section 4

3 HC versus $\mathbb{N}^{\mathbb{N}}$

3.1 Coding of HC

In this section we will consider the complexity of HC and the complexity of subsets of $\mathbb{N}^{\mathbb{N}}$ definable over HC. First we need a mechanism for coding elements of HC as elements in $\mathbb{N}^{\mathbb{N}}$.

Definition 9 We let C be the least subset of $\mathbb{N}^{\mathbb{N}}$ that satisfies

1. If $f(0) = 0$ then $f \in C$.
2. If $f = \langle k+1, \langle f_i \rangle_{i \in \mathbb{N}} \rangle$ and each $f_i \in C$ then $f \in C$.

C is defined by a positive induction, and we use the elements of C as codes for elements in HC using the decoding function ρ_C:

Definition 10 1. If $f(0) = 0$, let $\rho_C(f) = \emptyset$.

2. If $f = \langle k+1, \langle f_i \rangle_{i \in \mathbb{N}} \rangle$, let

$$\rho_C(f) = \{\rho_C(f_i) \mid i \in \mathbb{N}\}.$$

ρ_C is well defined on C, but our definition makes no sense outside of C. In a model for set theory where we may have non well founded sets, it is possible to consider ρ_C to be total.
ρ_C is also onto:

Lemma 5 *For each $x \in \mathsf{HC}$ there is an $f \in C$ such that $x = \rho_C(f)$.*

The proof is trivial by induction on the rank of x. Note that we actually need the countable axiom of choice in the proof.

Now, let $\vec{s} = (s_1, \dots, s_k)$ be a finite sequence from ω, where we let e denote the empty sequence, corresponding to $k = 0$.
Let $f \in \mathbb{N}^{\mathbb{N}}$. By recursion on the length of the sequence $\vec{s}$, we may define the *predecessor $f_{\vec{s}}$ of f with index $\vec{s}$* as follows:

- $f_e = f$.
- $f_{\vec{s}s}(n) = f_{\vec{s}}(1 + \langle s, n \rangle)$.

Lemma 6 *Let $f \in \mathbb{N}^{\mathbb{N}}$. The following are equivalent:*

1. $f \in C$.
2. $\forall g \exists n (f_{(g(0) \cdots g(n-1))}(0) = 0)$.

Proof
Since C is inductively defined, we can prove 1. $\Rightarrow$ 2. by induction on the formation of elements in C.
In order to prove the lemma in the other direction, we let T_f be the tree of seqences $\vec{s}$ such that we for no proper subsequence $\vec{t}$ have that $f_{\vec{t}}(0) = 0$. A reformulation of 2. is that T_f is well founded, i.e. has no infinite branches. Then we may use induction on the ordinal rank of T_f and prove that when T_f is well founded, then $f \in C$.

Corollary 4 *C is a Π^1_1-set.*

Our next task is to translate Δ_0 statements over HC to statements over C. The main obstacle will be to describe when two elements of C codes the same element in HC, and when this obstacle is taken care of, the rest is easy.

Lemma 7 *There is one Π^1_1-statement $\Pi_=$ and one Σ^1_1-statement $\Sigma_=$ such that whenever f and g are in C then*

$$\rho_C(f) = \rho_C(g) \Leftrightarrow \Sigma_=(f,g) \leftrightarrow \Pi_=(f,g).$$

Proof
Let V be a binary predicate on the set of finite sequences from $\mathbb{N}$. Given f and g we say that V is an identity predicate for f and g, $I(V,f,g)$, if for all sequences $\vec{s}$ and $\vec{t}$ we have that

$\vec{s}V\vec{t}$ if and only if either $f_{\vec{s}}(0) = g_{\vec{t}}(0) = 0$ or if

- $f_{\vec{s}}(0) > 0$
- $g_{\vec{t}}(0) > 0$
- $\forall s \exists t (\vec{s}sV\vec{t}t)$
- $\forall t \exists s (\vec{s}sV\vec{t}t)$

If $f \in C$ and $g \in C$, there will be exactly one identity predicate V for f and g, and then $\rho_C(f) = \rho_C(g)$ if and only if eVe (where e still is the empty sequence). Let

$$\Sigma_=(f,g) \Leftrightarrow \exists V((I(V,f,g) \wedge eVe)$$

and

$$\Pi_=(f,g) \Leftrightarrow \forall V(I(V,f,g) \rightarrow eVe).$$

This ends our proof, as we leave the rest of the details for the reader.

Exercise 4 Use Lemma 6 and the proof of Lemma 6 and prove

Let $\Phi(x_1, \dots, x_n)$ be Δ_0. Then there is a Σ^1_1 formula Σ_Φ and a Π^1_1-formula Π_Φ such that whenever $f_1, \dots, f_n$ are in C we have

$$\Phi(\rho_C(f_1), \dots, \rho_C(f_n)) \Leftrightarrow \Sigma_\Phi(f_1, \dots, f_n) \Leftrightarrow \Pi_\Phi(f_1, \dots, f_n).$$

Hint: Use the proof of Lemma 6 when Φ is $x_i \in x_j$ and combine Lemma 6 with induction on the complexity of Φ otherwise.

We will use this machinery to show that all Σ_1 definable subsets of $\mathbb{N}^\mathbb{N}$ actually are Σ^1_2.

Lemma 8 *There is a Δ^1_1-definable map $f \mapsto c_f$ such that whenever $f \in \mathbb{N}^\mathbb{N}$ then $c_f \in C$ and $\rho_C(c_f) = f$.*

Proof
Since f set-theoretically is a subset of $\omega \times \omega$ we first construct a code $g_{n,m}$ for $\langle n, m\rangle$ for each such pair, and then let

$$c_f = \langle 1, \langle g_{n,f(n)}\rangle_{n\in\mathbb{N}}\rangle.$$

The construction of $g_{n,m}$ is left for the reader.
The whole construction is computable and thus arithmetical and Δ^1_1.

Corollary 5 *If $A \subset \mathbb{N}^\mathbb{N}$ is Σ_1, then A is Σ^1_2.*

Proof
By Theorem 3 we may assume that

$$f \in A \Leftrightarrow \exists x \in \mathsf{HC}\Phi(x, f)$$

where Φ is Δ_0.
Then

$$f \in A \Leftrightarrow \exists g(g \in C \wedge \Pi_\Phi(g, c_f))$$

and this is Σ^1_2.

The converse will actually also hold, all Σ^1_2-sets are Σ_1-definable. This will be proved in Section 4, see 7.

3.2 $\mathsf{V} = \mathsf{L}$ and the well ordering of $\mathbb{N}^\mathbb{N}$

We have shown (see Kunen [1]) that if $\mathsf{V} = \mathsf{L}$, then the axion of choice will hold. We have also shown that the continuum hypothesis holds under the assumption of $\mathsf{V} = \mathsf{L}$. A consequence of these two results is

Lemma 9 *If $\mathsf{V} = \mathsf{L}$ then $\mathsf{HC} = \mathsf{L}(\aleph_1)$.*

We constructed the well ordering of L using iterated transfinite recursion, some Δ_1-definable constants and some Δ_1-definable functions. In the same way as we prove that $\mathsf{L}(\alpha)$, seen as a function, is Δ_1, we get that the function giving us the well ordering of $\mathsf{L}(\alpha)$ also is Δ_1.
We then get

Lemma 10 *If* $\mathsf{V} = \mathsf{L}$ *there is a* Δ_1 *well ordering of* V.

Corollary 6 *If* $\mathsf{V} = \mathsf{L}$ *there is a* Δ^1_2*-well ordering of* $\mathbb{N}^{\mathbb{N}}$.

The significance is that whenever we can use a well ordering of $\mathbb{N}^{\mathbb{N}}$, or of the reals, to construct a nasty set like a non measurable one, it is consistent with ZF that there are Δ^1_2-sets being nasty in the chosen way.

4 The Shoenfield absoluteness theorem

Π^1_1-sets and Σ^1_1-sets are defined with the use of function quantifiers, and a priori there is no reason to believe that such definitions are absolute with respect to sets or classes with few function quantifiers.
In this section we will show that all Π^1_1-sets are absolute for transitive models of a fairly weak fragment of set theory, and that all Σ^1_2-sets are absolute for all models of set theory containing all the ordinals. In particular, all Σ^1_2-definitions will be absolute for L.

Lemma 11 *If* $A \subseteq \mathbb{N}^{\mathbb{N}}$ *is* Π^1_1*, then* A *is* Σ_1*-definable.*

Proof
Recall the normal form theorem for Π^1_1-sets, see Theorem 5, and let

$$g \in A \leftrightarrow \forall f \exists n R(g, f, n)$$

where we only use bounded quantifiers and symbols for computable functions and relations in defining R.
As a consequence, it is possible to determine if $R(g, f, n)$ from a finite amount of information about f and g. Indeed, there is a computable binary relation R^* on the set of sequence numbers (for finite sequences from $\mathbb{N}$) such that for each f, g, n the following will hold

1. $R(g, f, n) \Rightarrow \exists m \geq n R^*(\bar{g}(m)\bar{f}(m))$
2. $R^*(\bar{g}(n), \bar{f}(n)) \Rightarrow \exists m \leq m R(g, f.m)$
3. $R^*(\bar{g}(n), \bar{f}(n)) \Rightarrow \forall m \geq n R^*(\bar{g}(m)\bar{f}(m))$

Here $\bar{f}(n)$ denotes the sequence number of the sequence $(f(0), \ldots, f(n-1))$, and when τ is a sequence of numberss, we will let $\#\tau$ denote the corresponding sequence number. Now, let $g \in \mathbb{N}^{\mathbb{N}}$ and let τ be a sequence of numbers of length n.
We let

$$\tau \in T_g \Leftrightarrow \neg R^*(\bar{g}(n), \#\tau).$$

T_g will be a tree of finite sequences from $\mathbb{N}$, and T will have no infinite branches if and only if $g \in A$. The map $g \mapsto T_g$ is computable, and in particular Δ_1.
In other words, $g \in A$ if and only if T_g is well founded and if and only if there is an order preserving function from T_g with the reversed sub sequence ordering to ON. The latter gives us the Σ_1–form.

Corollary 7 *All Σ^1_2-subsets of $\mathbb{N}^\mathbb{N}$ are Σ_1-definable.*

Proof
$\mathbb{N}^\mathbb{N}$ is itself Δ_1-definable (try to decide if it is actually Δ_0-definable), so an extra existential quantifier over $\mathbb{N}^\mathbb{N}$ cannot break a class out of Σ_1.

If M is a transitive model for a fragment of ZF strong enough to prove that well founded relations over $\mathbb{N}$ have rank functions, then all Π^1_1-definitions will be absolute for M. Such models are often called β-models. We will now extend this absoluteness observation to the Shoenfield absoluteness theorem.

Definition 11 Let A be Π^1_1, and for $g \in \mathbb{N}^\mathbb{N}$, let T_g be the tree we defined in the proof of Lemma 11.
For each ordinal α and for each function $g \in \mathbb{N}^\mathbb{N}$, we let $D_{\alpha,g}$ be a tree of finite sequences $(\alpha_0, \ldots, \alpha_{k-1})$ of ordinals $< \alpha$ as follows:
$(\alpha_0, \ldots, \alpha_{k-1}) \in D_{\alpha,g}$ if for all $i, j < k$, if $i = \#\sigma$ and $j = \#\tau$ for some σ and τ in T_g where σ is a proper extension of τ, then $\alpha_i < \alpha_j$.

The point is that $D_{\alpha,g}$ will contain finite sequences that locally looks like rank functions for T_g. Indeed, $D_{\alpha,g}$ will have an infinite branch if and only if T_g is well founded and has an ordinal rank $\leq \alpha$. This gives us

Lemma 12 *$g \notin A \leftrightarrow D_{\alpha,g}$ is well founded for all ordinals α.*

.

Lemma 13 *For all $g \in \mathsf{L}$ and ordinals α, the tree $D_{\alpha,g} \in \mathsf{L}$.*

Proof
The definition of $D_{\alpha,g}$ depends only on α and T_g, and the construction is absolute with respect to L.

Now, let $B \subseteq \mathbb{N}^\mathbb{N}$ be Π^1_2. We will see that membership in B also can be rephrased to the well foundedness of all trees in a class absolute for L. Observe that in order to decide if $(\alpha_0, \ldots, \alpha_{k-1})$ is in $D_{\alpha,g}$, we only need a finite approximation to g, we just need to be able to decide if $\sigma \in T_g$ whenever $\#\sigma < k$, and then $(g(0), \ldots, g(k-1))$ will suffice. We then say that $(\alpha_0, \ldots, \alpha_{k-1}) \in D_{\alpha,(g(0),\ldots,g(k-1))}$

Definition 12 Let $f \in B \leftrightarrow \forall h(\langle f, h\rangle \notin A)$ where A is Π^1_1 as above.
For $f \in \mathbb{N}^\mathbb{N}$ and $\alpha \in \mathsf{ON}$, let
$E_{\alpha,f} =$
$\{((a_0, \ldots, a_{k-1}), (\alpha_0, \ldots, \alpha_{k-1})) \mid (\alpha_0, \ldots, \alpha_{k-1}) \in D_{\alpha,(\langle f(0),a_0\rangle,\ldots,\langle f(k-1),a_{k-1}\rangle)}\}$.

The elements in $E_{\alpha,f}$ are pairs of sequences, and we order $E_{\alpha,f}$ using reversed pairwise sequence extension. An infinite branch in $E_{\alpha,f}$ will give a function h and an infinite branch in $D_{\alpha,\langle f,h\rangle}$, and conversely, each h and inn´finite branch in $D_{\alpha,\langle f,h\rangle}$ will give an infinite branch in $E_{\alpha,f}$. Thus we have
$f \in B \Leftrightarrow \forall h(\langle f, h\rangle \notin A) \Leftrightarrow \forall h \forall \alpha D_{\alpha,\langle f,h\rangle}$ is well founded $\Leftrightarrow \forall\alpha(E_{\alpha,f}$ is well

founded).
Further, we see that if $f \in \mathsf{L}$, then the definition of $E_{\alpha,f}$ is absolute, so $E_{\alpha,f} \in \mathsf{L}$. This gives us

Theorem 6 *All Π^1_2–statements are absolute for* L.

Proof
Since being well founded is absolute, the statement

$$(\forall \alpha \in \mathsf{ON}) E_{\alpha,f} \text{ is well founded}$$

be absolute for L. As we have seen, membership in a Π^1_2-set can be expressed this way.

Corollary 8 (The Shoenfield Absoluteness Theorem) *All Σ^1_2-definitions are absolute for all transitive models M of* ZF *such that* $\mathsf{ON} \subset M$.

References

[1] K. Kunen, Set Theory An introduction to Independence Proofs, Elsevier 1980.