Exam - MAT4701

The exam will be 40 minutes. It is divided into two parts as indicated below.

1 Presentation

Choose *one* of the following exercises to present in the first 15 minutes of the exam. Note the time limit. Please practice completing the presentation quickly (writing fast on the blackboard actually takes some practice).

Most of the exercises can be solved trivially by referring to a theorem from the curriculum. This is of course *not* how you are supposed to do the presentation. Instead you should modify the proofs to fit the (much simpler) setting.

1. Let $g \in C^2(\mathbb{R})$ and $\{B_t\}_{t\geq 0}$ be a 1-dimensional Brownain motion. Prove that
$$g(B_t) = g(0) + \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds.$$
2. Show that the Itô-integral is continuous in t, i.e. for $\varphi \in L_a^2([0,T] \times \Omega)$ show that there exists a continuous modification of the stochastic process
$$\left\{\int_0^t \varphi(s)dB_s\right\}_{t\geq 0}.$$
3. Assume $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz and Z is a random variable independent of the 1-dimensional Brownian motion $\{B_t\}_{t\geq 0}$. Prove that there exists a strong solution to
$$dX_t = \sigma(X_t)dB_t, \qquad X_0 = Z.$$
4. Show that
$$dX_t^x = \sigma(X_t^x)dB_t, \qquad X_0^x = x.$$
has a version that is continuous in x. You may take for granted that
$$E\left[\left(\int_0^t \varphi(s)dB_s\right)^{2p}\right] \leq CE\left[\int_0^t |\varphi(s)|^{2p}ds\right]$$
for an appropriate constant C.
5. Extend the notion of Itô-integrals to local martingales. I.e. assume φ is an adapted process such that $P(\int_0^T \varphi(s)^2 ds < \infty) = 1$. Define the Itô-integral $\int_0^t \varphi(s)dB_s$ using stopping times.

6. Assume $\{a(t)\}_{t\geq 0}$ is a bounded adapted stochastic process. Show that

$$X_t := \int_0^t a(s)ds + B_t$$

is a Brownian motion under an appropriate measure.

7. Assume that $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a measurable and bounded function. Show that there exists a weak solution to

$$dX_t = b(t, X_t)dt + dB_t, \qquad X_0 = 0.$$

2 Questions

The rest of the time I will ask you questions. Typical questions will be

- **What is ... ?**

  where ... can be any definition from the curriculum sheet (e.g. Brownain motion, martingale, filtration, weak solution).

- **What does ... say?**

  where ... can be any result from the curriculum sheet (e.g. Doob's maximal inequality, Girsanov's theorem, Martingale representation theorem)

- **How would you prove ... ?**

  where ... can be any result from the curriculum sheet (e.g. Dynkin's formula, Continuity of the Itô-integral, Martingale property of the Itô-integral)