The Local Property of the Itô integral

In this note we will prove that the Itô integral satisfies the following local property:

Lemma 0.1. *For* $\varphi \in L_a^2([0,T]\times\Omega)$ *we let* $G_\varphi := \left\{\omega \in \Omega : \int_0^T \varphi^2(s,\omega)ds = 0\right\}$.

Then

$$\int_0^T \varphi(s,\omega)dB_s(\omega) = 0 \qquad P-\textit{almost every } \omega \in G_\varphi.$$

Proof. Step 1: Assume that $\varphi \in \mathcal{E}$, i.e. φ is on the form

$$\varphi(s,\omega) = \sum_{j=1}^{m} \xi_j(\omega) 1_{[t_j,t_{j+1})}(s)$$

where ξ_j is $\mathcal{F}_{t_j}$-measurable and bounded. For $\omega \in G_\varphi$ we have that

$$\sum_{j=1}^{m} \xi_j^2(\omega)|t_{j+1} - t_j| = \int_0^T \varphi(s,\omega)^2 ds = 0$$

so that $\xi_j(\omega) = 0$ for all $j = 1,\dots,m$. Consequently we have, for $\omega \in G_\varphi$

$$\int_0^T \varphi(s,\omega)dB_s(\omega) = \sum_{j=1}^{m} \xi_j(\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)) = 0.$$

Thus the Lemma is true for $\varphi \in \mathcal{E}$.

Step 2: Assume φ is bounded and continuous. We approximate φ by

$$\varphi^n(s,\omega) = \sum_{j=1}^{2^n-1} \varphi(t_j,\omega) 1_{[t_j,t_{j+1})}(s)$$

where $t_j = 2^{-n}jT$.

For $\omega \in G_\varphi$ we have that

$$\int_0^T \varphi^2(s,\omega)ds = 0.$$

Since φ is continuous in s we must have that $\varphi(s,\omega) = 0$ for all $s \in [0,T]$ and consequently

$$\varphi^n(s,\omega) = 0.$$

We know that $\int_0^T \varphi^n(s)dB_s \to \int_0^T \varphi(s)dB_s$ in $L^2(\Omega)$ and from Step 1 we get that $\int_0^T \varphi^n(s,\omega)dB_s(\omega) = 0$ for all $\omega \in G_\varphi$ and thus the Lemma is true for φ bounded and continuous.

Step 3: Assume φ is bounded. We approximate φ by

$$\varphi^n(s,\omega) = \int_0^s \psi_n(r-s)\varphi(r,\omega)dr.$$

where ψ_n is the standard mollifier as used in class. For $\omega \in G_\varphi$ we have that $\int_0^T \varphi(s,\omega)^2 ds = 0$ so that $\varphi(s,\omega) = 0$ for almost all $s \in [0,T]$ and consequently $\varphi^n(s,\omega) = 0$ for all s.

We know that $\int_0^T \varphi^n(s)dB_s \to \int_0^T \varphi(s)dB_s$ in $L^2(\Omega)$ and from Step 2 we get that $\int_0^T \varphi^n(s,\omega)dB_s(\omega) = 0$ for all P-almost every $\omega \in G_\varphi$ and thus the Lemma is true for φ bounded.

Step 4 Assume $\varphi \in L^2_a([0,T]\times\Omega)$. We approximate φ by

$$\varphi^n(s,\omega) = \begin{cases} \varphi(s,\omega) & \text{if } |\varphi(s,\omega)| \le n \\ 0 & \text{if } |\varphi(s,\omega)| > n \end{cases}.$$

For all $\omega \in G_\varphi$ we have that

$$\int_0^T \varphi^2(s,\omega)ds = 0$$

so that $\varphi(s,\omega) = 0$ for almost all $s \in [0,T]$. Consequently $\varphi^n(s,\omega) = \varphi(s,\omega)$ for almost all $s \in [0,T]$, and so $\int_0^T \varphi^n(s,\omega)^2 ds = 0$.

We know that $\int_0^T \varphi^n(s)dB_s \to \int_0^T \varphi(s)dB_s$ in $L^2(\Omega)$ and from Step 3 we get that $\int_0^T \varphi^n(s,\omega)dB_s(\omega) = 0$ for P-a.e. $\omega \in G_\varphi$ and thus the Lemma is true for $\varphi \in L^2_a([0,T]\times\Omega)$.

□

Although the result seems obvious, the proof requires surprisingly many details. More specifically, we go through the steps of the construction of the Itô integral like we did in class, and show that the result holds for all the steps.

It could be tempting to use the Itô isometry as follows:

$$\begin{aligned} E[1_{G_\varphi}(\int_0^T \varphi(s)dB_s)^2] = E[(1_{G_\varphi}\int_0^T \varphi(s)dB_s)^2] &= E[(\int_0^T 1_{G_\varphi}\varphi(s)dB_s)^2] \\ = E[\int_0^T (1_{G_\varphi}\varphi(s))^2 ds] &= E[1_{G_\varphi}\int_0^T \varphi^2(s)ds] \\ &= 0 \end{aligned}$$

since $1_{G_\varphi}(\omega)\int_0^T \varphi^2(s,\omega)ds = 0$ for all $\omega \in \Omega$. Notice that this is *not* correct, since in the second equality in the first line we have used

$$1_{G_\varphi}\int_0^T \varphi(s)dB_s = \int_0^T 1_{G_\varphi}\varphi(s)dB_s.$$

But this is not true! Indeed, 1_{G_φ} is in general $\mathcal{F}_T$ measurable, so that the above integrand is not adapted and we cannot construct the Itô integral for such integrands.

Corollary 0.2. *Let* $\theta, \psi \in L_a^2([0,T]\times\Omega)$ *and let* $A := \left\{\omega \in \Omega : \int_0^T (\theta(s,\omega) - \psi(s,\omega))^2 ds = 0\right\}$. *Then*

$$\int_0^T \theta(s,\omega) dB_s(\omega) = \int_0^T \psi(s,\omega) dB_s(\omega)$$

for P*-almost all* $\omega \in A$.

Proof. Let $\varphi(s,\omega) := \theta(s,\omega) - \psi(s,\omega)$. Then

$$\int_0^T \varphi(s,\omega) dB_s(\omega) = 0$$

for almost all $\omega \in G_\varphi$ as in the previous Lemma. The result follows from the linearity of the Itô integral and the fact that $A = G_\varphi$.

□