

Voluntary exercise:

For $M, N \in \mathcal{M}_2^c$, define the cross-variation

$$\langle M, N \rangle_t := \frac{1}{4} [\langle M+N \rangle_t - \langle M-N \rangle_t]$$

1. Show that $MN_t - \langle M, N \rangle_t$ is a martingale.

2. Show that $\langle M, N \rangle_t = \lim_{m \rightarrow \infty} \sum_{n=1}^m (M_{t_n} - M_{t_{n-1}})(N_{t_n} - N_{t_{n-1}})$ in probability.

Hint: Go through the proof that

$$\langle M \rangle_t = \lim_{m \rightarrow \infty} \sum_{n=1}^m (M_{t_n} - M_{t_{n-1}})^2$$

and follow the same steps.

3. Prove that for $\varphi, \psi \in L_a([0, T] \times \Omega)$, $M_t := \int_0^t \varphi(s) dB_s$, $N_t := \int_0^t \psi(s) dB_s$

$$\langle M, N \rangle_t = \int_0^t \varphi(s) \psi(s) ds \quad P\text{-a.s.}$$

in two different ways:

i) Show the convergence from in 2.

ii) Use Itô's Formula to show $MN_t - \int_0^t \varphi(s) \psi(s) ds$ is a martingale

4. For $\varphi, \psi \in L_a([0, T] \times \Omega)$, $B^{(1)}, B^{(2)}$ indep. B.M. show

$$M_t = \int_0^t \varphi(s) dB_s^{(1)}, \quad N_t = \int_0^t \psi(s) dB_s^{(2)} \quad \text{we have}$$

$\langle M, N \rangle_t = 0$ in the same two different ways as in 3