MAT4701 Stochastic Analysis and Applications
Summary of Program

References

Content

Preliminaries

1. Elements of measure theory and probability spaces: definition of sigma-algebra, definition of measures, measurable space, probability space, P-completeness, P-augmentation.


5. Relationships among the various notions of convergence.

6. Absolutely continuous measures and Radon-Nikoym theorem. Connection with absolutely continuous distributions (w.r.t. Lebesgue).

Ref: [A] Chapter 1 pages 1-9, Section 1.1.5; [Ø] Section 2.1; Notes on web about convergences.

7. Conditional expectation: definition, properties (existence is studied via Radon-Nykodim derivative)

8. Independence and product measures

Ref: [A] Sections 1.1.3, 1.1.4, [Ø] Appendix B


Ref: [A] Sections 1.1.6 (Theorem 1.1.13 was not introduced). [Ø] Appendix A
Infinitely divisible distributions

10. Convolution of two measures, convolution of two probability distributions and its interpretation as probability distribution of the sum of two independent random variables. From 2 to any number of measures/probability distributions/random variables.

Reference [A] Section 1.2.1 (no proofs), Corollary 1.2.3 with proof.

11. Use of the characteristic function to study the sum of two independent random variables (exploit the one-to-one relationship between characteristic function of a random variable and probability distribution)

12. Definition of infinite divisible distribution and characterization (Prop 1.2.6)
13. Examples: Poisson distribution, the compound Poisson type distribution, the Gaussians.

Reference [A] Sections 1.2.2, 1.2.3

14. Levy-Khintchine theorem: the representation of the characteristic function of an infinitely divisible distribution, hence the characterization of infinitely divisible distributions (Main theorem: comments on the structure of the proof only). The characteristic triplet of the distribution

Reference [A] Section 1.2.3. The section present the terminology of Levy symbol, which we have seen and used as a concept, though we have not used specifically this terminology.

Stochastic processes

15. Stochastic processes, definition, paths/trajectories,
16. modification/version, indistinguishable processes, measurable version (sufficient condition and full characterization)
17. Finite dimensional distributions, consistency, Kolmogorov extension theorem.
19. Continuous version and Kolmogorov continuity theorem
20. Cadlag version

References [A] Section 1.1.7, [Ø] Chapter 2
As for measurability, specific additional references:
• J.L. Doob: Stochastic Processes, Wiley 1953
21. Levy processes
22. Distribution of any random variable of a Levy process is infinitely
    divisible (work with characteristic functions); the characteristic function
    of any random variable of a Levy process is explicitly given by the
    characteristic exponent of one of them.
23. Examples of Levy processes: Brownian motion, Poisson process,
    Compound Poisson process

Reference [A] Section 1.3: Prop 1.3.1, Thm 1.3.1 with proof. Section 1.3.1.
Statement of Corollary 1.4.6

Stochastic processes and information

24. Filtrations as model of information: usual hypotheses on the filtration
25. Adapted processes
26. Martingales (sub-, super-)
27. Closed martingales
28. Martingales with values in $L^2$
29. Results (no proof): Doob martingale inequality, Doob tail martingale
    inequality

Reference [A] Section 2.1: as for Theorem 2.1.8, 2.1.10 (no proof); Section 2.1.3

30. Stopping times, stopped processes (localization), stopped sigma-algebras
    (optional sigma-algebras), Doob optional sampling theorem (Doob
    optional stopping theorem)
31. Local martingale
32. Semimartingale

References:
[A] Sections 2.1.2, 2.2, 2.7

33. Doob--Meyer decomposition, angle brackets.
34. Lévy characterization of Brownian motion.

References:
[A] Section 2.2.1 (no Doob-Meyer 1)

35. The jumps of a Lévy process and the Poisson random measure.
36. Poisson integration.

References:
[A] Section 2.3 (no proofs, but look at the representation of the measure N also
    contained in the proof of Lemma 2.3.4), Sections 2.3.1, 2.3.2

37. Processes of finite variation

Ref [A] Sections 2.3.3 pages 110-112
38. Lévy-Itô decomposition

Ref [A] Section 2.4: of this section the statements Theorems 2.4.7, 2.4.11, 2.4.15, 2.4.16. Corollary 2.4.20, Note 2, Corollary 2.4.21, Theorem 2.4.25

Stochastic integration

39. Motivations from Brownian motion
40. Infinite total variation of Brownian motion (proof) and nowhere differentiability of Brownian motion

Ref [Karatzas and Shreve: Brownian motion and stochastic calculus, Springer] pages 109-110

41. Finite quadratic variation of Brownian motion: Lévy theorem (L^2-convergence with proof and P-a.s. convergence only statement)
42. Itô stochastic integration for adapted (predictable) integrands: construction, Itô isometry

Reference [Ø] Chapter 3

43. Martingale random fields (real-valued independently scattered martingale-valued measures as Applebaum calls them).
44. Examples
45. Predictable sigma-algebra and predictable processes, integrands
46. Construction of the Itô stochastic integral (L^2 framework)
47. Itô isometry
48. Properties of the stochastic integral in L^2

Reference: [A] Section 4.1, 4.2.1 No proof of lemma 4.1.4

49. Construction of the extended Itô integral
50. Properties
51. Examples

Ref [A] Section 4.2.2, 4.2.3, 4.3.3 Proof of the Gihman-Skorohod lemma (Lemma 4.2.8). No proof Thm 4.2.12

Ito formula

52. The Ito formula is introduced first for Brownian motion with structure of the proof in steps and arguments details only for the first step.

Ref [Ø] Section 4.1
53. The Ito formula for Levy processes is introduced first for the Poisson random measure (Ref [A Lemma 4.4.5] with proof) and the for a general so-called Ito-Levy process (Ref [A Theorem 4.4.7, no proof only some comments]). Another general formula has been presented (Ref [A Theorem 4.4.10 no proof]).

54. Multidimensional Ito formula (statement)

55. Product rule

**Stochastic differential equations**

56. Stochastic Differential Equations (SDE) with jumps: Existence and uniqueness result

57. SDE driven by Levy processes

58. Linear case for Brownian motion

Reference [A] Section 6.2, for the Brownian case [Ø] Sections 5.1, 5.2; [A] Section 6.3; Notes about solution to linear differential equations in the Brownian setting

59. Stochastic exponential, Exponential martingales

Reference [A] Section 6.3, Section 5.2.2

59. Change of measure

60. Girsanov theorem Brownian case with proof (use of the Levy characterization of Brownian motion – with local martingales see [ØS])

61. Girsanov theorem for jumps no proofs

Reference [A] Section 5.2.3; [ØS] Section 1.4


[Ø] Section 8.6

62. Stochastic integral representation theorem

63. Martingale representation theorem in the $L^2$ setting

64. Local martingale representation theorem (discussed the statement in class, not in the program)

Reference [Ø] Section 4.3, [A] Theorems 5.3.5, 5.3.6