

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4790/9790 — Stochastic Filtering

Day of examination: Monday 4, December 2023

Examination hours: 9:00 AM – 13:00 AM

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Discrete time linear filtering.

a (weight 10p)

Describe the setup for the Kalman-Bucy filter and state the main result (No proof).

b (weight 10p)

Consider a pair of real valued processes $(X_j, Y_j)_{j \geq 0}$ in L^2 and let $\mathcal{L}_j^Y := \overline{\text{span}} \{1, Y_0, \dots, Y_j\}$. Prove that for $j \geq 1$ we have that

$$\hat{X}_j = \hat{X}_{j|j-1} + P_{j|j-1}^{XY} (P_{j|j-1}^Y)^{\oplus} (Y_j - \hat{Y}_{j|j-1}),$$

where

$$\begin{aligned} \hat{X}_j &:= \mathbb{E}[X_j | \mathcal{L}_j^Y], & \hat{X}_{j|j-1} &:= \mathbb{E}[X_j | \mathcal{L}_{j-1}^Y], & \hat{Y}_{j|j-1} &:= \mathbb{E}[Y_j | \mathcal{L}_{j-1}^Y], \\ P_{j|j-1}^{XY} &:= \mathbb{E}[(X_j - \hat{X}_{j|j-1})(Y_j - \hat{Y}_{j|j-1})], & P_{j|j-1}^Y &:= \mathbb{E}[(Y_j - \hat{Y}_{j|j-1})^2]. \end{aligned}$$

Hint: You can use, without proof, the orthogonal projection theorem.

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c (weight 10p)

Find the recursions for $\hat{X}_j$, the best linear prediction of X_j given $\mathcal{L}_j^Y := \overline{\text{span}} \{1, Y_0, \dots, Y_j\}$, and its error $P_j := \mathbb{E}[(X_j - \hat{X}_j)^2]$ when

$$\begin{aligned} X_j &= aX_{j-1} + bY_j + \varepsilon_j, & j \geq 1 \\ Y_j &= cX_j + dY_{j-1} + \xi_j, & j \geq 1, \end{aligned}$$

where $\{\varepsilon\}_{j \geq 1}$ and $\{\xi\}_{j \geq 1}$ are two independent sequences of orthogonal real valued white noises, $X_0 = Y_0 = 0$, and $bc \neq 1$.

Problem 2

Discrete time nonlinear filtering

a (weight 10p)

Describe the setup for finding the nonlinear filter via the Bayes formula and state the main result.

b (weight 10p)

Sketch the proof of the result in the previous subsection. If some auxiliary lemmas are needed you can state them without proof.

c (weight 10p)

Let $X_j = \varphi(X_{j-1})\varepsilon_j$ and $Y_j = X_j + \xi_j$ where φ is a Borel measurable function, bounded, and bounded away from 0 ($\varphi(x) \geq c > 0$). The sequences $(\varepsilon_j)_{j \geq 1}$ and $(\xi_j)_{j \geq 1}$ are independent of each other and i.i.d.. Moreover, ε_1 and ξ_1 have densities p and q with respect to the Lebesgue measure (and enough integrability of its moments). Find a recursion for the random measure $\pi_j(dx)$ satisfying

$$\int_{\mathbb{R}} f(x) \pi_j(dx) = \mathbb{E}[f(X_j) | \mathcal{F}_j^Y].$$

Problem 3

Continuous time linear filtering

(Noisy observations of a population growth) Consider a simple growth model

$$dX_t = rX_t dt, \quad \mathbb{E}[X_0] = b > 0, \quad \mathbb{E}[(X_0 - b)^2] = a^2,$$

for some constants $r > 0, b > 0$ and a . Suppose we observe Y_t satisfying

$$dY_t = X_t dt + m dV_t, \quad Y_0 = 0,$$

with $m > 0$ being a constant and V a one dimensional Brownian motion independent of X_0 .

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a (weight 10p)

Find the equations for $\widehat{X}_t$, the best linear estimator of X_t using the values of the process Y up to time t , and $P_t = \mathbb{E}[(X_t - \widehat{X}_t)^2]$.

b (weight 10p)

Solve the equation for $\widehat{X}_t$. You can leave the solution in terms of P_t , that is, you do not need to solve the Riccati equation for P_t .

Problem 4

Continuous time nonlinear filtering

a (weight 10p)

Describe the general setup for the nonlinear filtering problem in continuous time and the main result (No proof).

b (weight 10p)

State and sketch the proof of the Kushner-Stratonovich equation. Introduce the processes involved and the conditions assumed. If some auxiliary lemmas are needed you can state them without proof.