

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT4750/9750 — Mathematical Finance:
Modelling and risk management

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This problem set consists of 12 pages.

Appendices: None

Permitted aids: All

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

We consider a pair of processes $(X, Y) = (X_j, Y_j)_{j \geq 0}$, generated by the linear recursive equations

$$X_j = a_0(j) + a_1(j) X_{j-1} + a_2(j) Y_{j-1} + b_1(j) \varepsilon_j + b_2(j) \xi_j, \quad (1)$$

$$Y_j = A_0(j) + A_1(j) X_{j-1} + A_2(j) Y_{j-1} + B_1(j) \varepsilon_j + B_2(j) \xi_j, \quad (2)$$

where

- X_j and Y_j has values in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.
- $\varepsilon = (\varepsilon_j)_{j \geq 1}$ and $(\xi_j)_{j \geq 1}$ are orthogonal discrete time white noises with values in $\mathbb{R}^l$ and $\mathbb{R}^k$. That is,

$$\begin{aligned} \mathbb{E}[\varepsilon_j] &= 0, & \mathbb{E}[\varepsilon_j \varepsilon_i^T] &= \begin{cases} I, & i = j \\ 0, & i \neq j \end{cases} \in \mathbb{R}^{l \times l}, \\ \mathbb{E}[\xi_j] &= 0, & \mathbb{E}[\xi_j \xi_i^T] &= \begin{cases} I, & i = j \\ 0, & i \neq j \end{cases} \in \mathbb{R}^{k \times k}, \end{aligned}$$

and $\mathbb{E}[\varepsilon_j \xi_i^T] = 0$, for all $i, j \geq 0$.

- The coefficients $a_0(j)$, $a_1(j)$, etc. are deterministic (known) sequences of matrices of appropriate dimensions. In what follows we will drop the dependence of the coefficients on time, to lighten the notation.

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- The equations are solved subject to possibly random initial conditions X_0 and Y_0 , uncorrelated with the noises ε and ξ , whose means and covariances are known.

We denote the optimal linear estimate of X_j given $\mathcal{L}_j^Y = \overline{\text{span}}\{1, Y_1, \dots, Y_j\}$ by $\hat{X}_j = E[X_j | \mathcal{L}_j^Y]$ and the corresponding error covariance matrix by

$$P_j = \mathbb{E} \left[\left(X_j - \hat{X}_j \right) \left(X_j - \hat{X}_j \right)^T \right].$$

Theorem 1. *The estimate $\hat{X}_j$ and the error covariance P_j satisfy the equations*

$$\begin{aligned} \hat{X}_j &= a_0 + a_1 \hat{X}_{j-1} + a_2 Y_{j-1} \\ &\quad + (a_1 P_{j-1} A_1^T + b \circ B) (A_1 P_{j-1} A_1^T + B \circ B)^\oplus (Y_j - A_0 - A_1 \hat{X}_{j-1} - A_2 Y_{j-1}), \end{aligned} \quad (3)$$

and

$$\begin{aligned} P_j &= a_1 P_{j-1} a_1^T + b \circ b \\ &\quad - (a_1 P_{j-1} A_1^T + b \circ B) (A_1 P_{j-1} A_1^T + B \circ B)^\oplus (a_1 P_{j-1} A_1^T + b \circ B)^T, \end{aligned} \quad (4)$$

where

$$b \circ b = b_1 b_1^T + b_2 b_2^T, \quad b \circ B = b_1 B_1^T + b_2 B_2^T, \quad B \circ B = B_1 B_1^T + B_2 B_2^T,$$

and

$$\begin{aligned} \hat{X}_0 &= \mathbb{E}[X_0] + \text{cov}(X_0, Y_0) \text{cov}(Y_0)^\oplus (Y_0 - \mathbb{E}[Y_0]), \\ P_0 &= \text{cov}(X_0) - \text{cov}(X_0, Y_0) \text{cov}(Y_0)^\oplus \text{cov}(X_0, Y_0)^T. \end{aligned}$$

b (weight 10p)

To verify

$$\hat{X}_j = \hat{X}_{j|j-1} + P_{j|j-1}^{XY} (P_{j|j-1}^Y)^\oplus (Y_j - \hat{Y}_{j|j-1}),$$

we can check that

$$\eta := X_j - \hat{X}_{j|j-1} + P_{j|j-1}^{XY} (P_{j|j-1}^Y)^\oplus (Y_j - \hat{Y}_{j|j-1})$$

is orthogonal to $\mathcal{L}_j^Y$. Since $X_j - \hat{X}_{j|j-1}$ and $Y_j - \hat{Y}_{j|j-1}$ are orthogonal to $\mathcal{L}_{j-1}^Y$, then η is orthogonal to $\mathcal{L}_{j-1}^Y$. Since any $Z \in \mathcal{L}_j^Y$ can be written as

$$Z = a_0 + a_1 Y_j + \dots + a_{j-1} Y_{j-1} + a Y_j,$$

we have that $\langle \eta, Z \rangle = a_j \langle \eta, Y_j \rangle$ (because $a_0 + a_1 Y_j + \dots + a_{j-1} Y_{j-1} \in \mathcal{L}_{j-1}^Y$) and, therefore, η is orthogonal to $\mathcal{L}_j^Y$ if $\eta \perp Y_j$. Moreover, by the orthogonal projection theorem we have

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the decomposition $Y_j = \widehat{Y}_{j|j-1} + Y_j - \widehat{Y}_{j|j-1}$ with $\widehat{Y}_{j|j-1} \in \mathcal{L}_{j-1}^Y$ and $Y_j - \widehat{Y}_{j|j-1} \in \left(\mathcal{L}_{j-1}^Y\right)^\perp$. Therefore,

$$0 = \langle \eta, Y_j \rangle = \langle \eta, \widehat{Y}_{j|j-1} \rangle + \langle \eta, Y_j - \widehat{Y}_{j|j-1} \rangle = \langle \eta, Y_j - \widehat{Y}_{j|j-1} \rangle.$$

That is, $\eta \perp Y_j$ iff $\eta \perp Y_j - \widehat{Y}_{j|j-1}$. Then,

$$\begin{aligned} \mathbb{E} \left[\eta \left(Y_j - \widehat{Y}_{j|j-1} \right) \right] &= \mathbb{E} \left[\left(X_j - \widehat{X}_{j|j-1} + P_{j|j-1}^{XY} (P_{j|j-1}^Y)^\oplus (Y_j - \widehat{Y}_{j|j-1}) \right) \left(Y_j - \widehat{Y}_{j|j-1} \right) \right] \\ &= P_{j|j-1}^{XY} - P_{j|j-1}^{XY} (P_{j|j-1}^Y)^\oplus P_{j|j-1}^Y = 0. \end{aligned}$$

Note that, if $P_{j|j-1}^Y$ is a strictly positive constant $(P_{j|j-1}^Y)^\oplus P_{j|j-1}^Y = 1$ and if $P_{j|j-1}^Y = 0$ we have that $(P_{j|j-1}^Y)^\oplus P_{j|j-1}^Y = 0$ and $P_{j|j-1}^{XY} = 0$ because $Y_j - \widehat{Y}_{j|j-1}$ is a constant P-a.s.

c (weight 10p)

We have the system

$$\begin{aligned} X_j &= aX_{j-1} + bY_j + \varepsilon_j, & j \geq 1 \\ Y_j &= cX_j + dY_{j-1} + \xi_j, & j \geq 1, \end{aligned}$$

where $\{\varepsilon\}_{j \geq 1}$ and $\{\xi\}_{j \geq 1}$ are two independent sequences of orthogonal real valued white noises, $X_0 = Y_0 = 0$, and $bc \neq 1$. We first rewrite the equation for X_j as follows

$$\begin{aligned} X_j &= aX_{j-1} + bY_j + \varepsilon_j = aX_{j-1} + b(cX_j + dY_{j-1} + \xi_j) + \varepsilon_j \\ &= aX_{j-1} + bcX_j + bdY_{j-1} + b\xi_j + \varepsilon_j, \end{aligned}$$

and solving for X_j we get

$$X_j = \frac{a}{1-bc} X_{j-1} + \frac{bd}{1-bc} Y_{j-1} + \frac{b}{1-bc} \xi_j + \frac{1}{1-bc} \varepsilon_j$$

and substituting this expression in the equation for Y_j we get

$$\begin{aligned} Y_j &= cX_j + dY_{j-1} + \xi_j \\ &= \frac{ac}{1-bc} X_{j-1} + \frac{bcd}{1-bc} Y_{j-1} + \frac{bc}{1-bc} \xi_j + \frac{c}{1-bc} \varepsilon_j + dY_{j-1} + \xi_j \\ &= \frac{ac}{1-bc} X_{j-1} + \frac{d}{1-bc} Y_{j-1} + \frac{1}{1-bc} \xi_j + \frac{c}{1-bc} \varepsilon_j. \end{aligned}$$

Hence, the system can be rewritten as

$$X_j = \frac{a}{1-bc} X_{j-1} + \frac{bd}{1-bc} Y_{j-1} + \frac{b}{1-bc} \xi_j + \frac{1}{1-bc} \varepsilon_j,$$

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$$Y_j = \frac{ac}{1-bc}X_{j-1} + \frac{d}{1-bc}Y_{j-1} + \frac{1}{1-bc}\xi_j + \frac{c}{1-bc}\varepsilon_j$$

and, using the notation of the Kalman-Bucy filter, we have

$$\begin{aligned} a_0 &= 0, & a_1 &= \frac{a}{1-bc}, & a_2 &= \frac{bd}{1-bc}, & b_1 &= \frac{b}{1-bc}, & b_2 &= \frac{1}{1-bc}, \\ A_0 &= 0, & A_1 &= \frac{ac}{1-bc}, & A_2 &= \frac{d}{1-bc}, & B_1 &= \frac{1}{1-bc}, & B_2 &= \frac{c}{1-bc}, \end{aligned}$$

and

$$b \circ b = \frac{b^2 + 1}{(1-bc)^2}, \quad b \circ B = \frac{b+c}{(1-bc)^2}, \quad B \circ B = \frac{1+c^2}{(1-bc)^2},$$

and

$$a_1 P_{j-1} A_1^T = \frac{a^2 c}{(1-bc)^2} P_{j-1}, \quad A_1 P_{j-1} A_1^T = \frac{a^2 c^2}{(1-bc)^2} P_{j-1}, \quad a_1 P_{j-1} a_1^T = \frac{a^2}{(1-bc)^2} P_{j-1}.$$

Therefore,

$$\begin{aligned} \hat{X}_j &= \frac{a}{1-bc} \hat{X}_{j-1} + \frac{bd}{1-bc} Y_{j-1} \\ &\quad + \left(\frac{(a^2+1)c+b}{(1-bc)^2} P_{j-1} \right) \left(\frac{(a^2+1)c^2+1}{(1-bc)^2} \right)^\oplus \left(Y_j - \frac{ac}{1-bc} \hat{X}_{j-1} - \frac{d}{1-bc} Y_{j-1} \right), \end{aligned}$$

and

$$\begin{aligned} P_j &= \frac{a^2}{(1-bc)^2} P_{j-1} + \frac{b^2+1}{(1-bc)^2} \\ &\quad - \left(\frac{(a^2+1)c+b}{(1-bc)^2} P_{j-1} \right)^2 \left(\frac{(a^2+1)c^2+1}{(1-bc)^2} \right)^\oplus, \end{aligned}$$

Note that, since $(a^2+1)c^2+1 \neq 0$, we have

$$\left(\frac{(a^2+1)c^2+1}{(1-bc)^2} \right)^\oplus = \left(\frac{(a^2+1)c^2+1}{(1-bc)^2} \right)^{-1} = \frac{(1-bc)^2}{(a^2+1)c^2+1}$$

and we can write

$$\begin{aligned} \hat{X}_j &= \frac{a}{1-bc} \hat{X}_{j-1} + \frac{bd}{1-bc} Y_{j-1} + \left(\frac{(a^2+1)c+b}{(a^2+1)c^2+1} P_{j-1} \right) \left(Y_j - \frac{ac}{1-bc} \hat{X}_{j-1} - \frac{d}{1-bc} Y_{j-1} \right), \\ P_j &= \frac{b^2+1+a^2 P_{j-1}}{(1-bc)^2} - \frac{((a^2+1)c+b)^2}{((a^2+1)c^2+1)(1-bc)^2} P_{j-1}^2. \end{aligned}$$

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Problem 2

a (weight 10p)

Let $(X_j, Y_j)_{j \geq 0}$ be a pair of random sequences with the following structure:

- X_j is a Markov process with transition kernel $\Lambda(x, du)$ and initial distribution $p(du)$, that is, for all $B \in \mathcal{B}(\mathbb{R})$

$$P(X_j \in B | \mathcal{F}_{j-1}^X \vee \mathcal{F}_{j-1}^Y) = \int_B \Lambda(X_{j-1}, du), \quad P\text{-a.s.}$$

where $\mathcal{F}_{j-1}^X = \sigma(X_0, X_1, \dots, X_{j-1})$, $\mathcal{F}_{j-1}^Y = \sigma(Y_0, Y_1, \dots, Y_{j-1})$ and

$$P(X_0 \in B) = \int_B p(du).$$

- Y_j is a random sequence such that for all $j \geq 0$

$$P(Y_j \in B | \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y) = \int_B \Gamma(X_j, du), \quad P\text{-a.s.}$$

with a Markov kernel $\Gamma(x, du)$, which has density $\gamma(x, u)$ with respect to some σ -finite measure $\nu(du)$ on $\mathcal{B}(\mathbb{R})$.

Let $\pi_j(dx)$ be the solution of the recursive equation

$$\pi_j(dx) = \frac{\int_{\mathbb{R}} \gamma(x, Y_j) \Lambda(u, dx) \pi_{j-1}(du)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(x, Y_j) \Lambda(u, dx) \pi_{j-1}(du)} \quad j \geq 1,$$

subject to $\pi_0(dx) = \gamma(x, Y_0) p(dx) / (\int_{\mathbb{R}} \gamma(x, Y_0) p(dx))$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel measurable function such that $\mathbb{E}[|f(X_j)|] < +\infty$ for each $j \geq 0$, then

$$\mathbb{E}[f(X_j) | \mathcal{F}_j^Y] = \int_{\mathbb{R}} f(x) \pi_j(dx), \quad P\text{-a.s.}$$

b (weight 10p)

The regular conditional distribution of $(Y_0, Y_1, \dots, Y_j)$ given $\mathcal{F}_j^X$ is given by

$$\begin{aligned} & P(Y_0 \in A_0, \dots, Y_j \in A_j |) \\ &= \mathbb{E} \left[\prod_{k=0}^{j-1} \mathbf{1}_{\{Y_k \in A_k\}} P(Y_j \in A_j | \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y) \middle| \mathcal{F}_j^X \right] \\ &= \mathbb{E} \left[\prod_{k=0}^{j-1} \mathbf{1}_{\{Y_k \in A_k\}} \middle| \mathcal{F}_j^X \right] \int_{A_j} \gamma(X_j, u_j) \nu(du_j) \end{aligned}$$

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$$\begin{aligned}
&= \mathbb{E} \left[\prod_{k=0}^{j-1} \mathbf{1}_{\{Y_k \in A_k\}} \middle| \mathcal{F}_{j-1}^X \right] \int_{A_j} \gamma(X_j, u_j) \nu(du_j) \\
&= \mathbb{E} \left[\prod_{k=0}^{j-2} \mathbf{1}_{\{Y_k \in A_k\}} P(Y_{j-1} \in A_{j-1} | \mathcal{F}_{j-1}^X \vee \mathcal{F}_{j-2}^Y) \middle| \mathcal{F}_{j-1}^X \right] \int_{A_j} \gamma(X_j, u_j) \nu(du_j) \\
&= \mathbb{E} \left[\prod_{k=0}^{j-2} \mathbf{1}_{\{Y_k \in A_k\}} \middle| \mathcal{F}_{j-1}^X \right] \int_{A_{j-1}} \int_{A_j} \gamma(X_{j-1}, u_{j-1}) \gamma(X_j, u_j) \nu(du_{j-1}) \nu(du_j) \\
&\vdots \\
&= \int_{A_0} \cdots \int_{A_j} \prod_{k=0}^j \gamma(X_k, u_k) \nu(du_0) \cdots \nu(du_j).
\end{aligned}$$

Then using the multidimensional version of the Bayes formula for vectors, written in expectation form, allows us to write

$$\mathbb{E} [\varphi(X_j) | \mathcal{F}_j^Y] (\omega) = \frac{\dot{\mathbb{E}} [\varphi(X_j(\dot{\omega})) \prod_{k=0}^j \gamma(X_k(\dot{\omega}), Y_k(\omega))]}{\dot{\mathbb{E}} [\prod_{k=0}^j \gamma(X_k(\dot{\omega}), Y_k(\omega))]},$$

where the dot denotes that we are working on a copy $(\dot{\Omega}, \dot{\mathcal{F}}, \dot{P})$ of the original probability space $(\Omega, \mathcal{F}, P)$ and the X_j are considered random variables in that space. We introduce the notation

$$L_j(X(\dot{\omega}), Y) := \prod_{k=0}^j \gamma(X_k(\dot{\omega}), Y_k)$$

and realize that

$$\begin{aligned}
&\dot{\mathbb{E}} \left[\varphi(X_j(\dot{\omega})) \prod_{k=0}^j \gamma(X_k(\dot{\omega}), Y_k) \middle| \mathcal{F}_{j-1}^X \right] \\
&= L_{j-1}(X(\dot{\omega}), Y) \dot{\mathbb{E}} [\varphi(X_j(\dot{\omega})) \gamma(X_j(\dot{\omega}), Y_j) | \mathcal{F}_{j-1}^X] \\
&= L_{j-1}(X(\dot{\omega}), Y) \int_{\mathbb{R}} \varphi(u) \gamma(u, Y_j) \Lambda(X_{j-1}(\dot{\omega}), du).
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E} [\varphi(X_j) | \mathcal{F}_j^Y] &= \frac{\dot{\mathbb{E}} [\varphi(X_j(\dot{\omega})) \prod_{k=0}^j \gamma(X_k(\dot{\omega}), Y_k)]}{\dot{\mathbb{E}} [\prod_{k=0}^j \gamma(X_k(\dot{\omega}), Y_k)]} \\
&= \frac{\dot{\mathbb{E}} [L_{j-1}(X(\dot{\omega}), Y) \int_{\mathbb{R}} \varphi(u) \gamma(u, Y_j) \Lambda(X_{j-1}(\dot{\omega}), du)]}{\dot{\mathbb{E}} [L_{j-1}(X(\dot{\omega}), Y) \int_{\mathbb{R}} \gamma(u, Y_j) \Lambda(X_{j-1}(\dot{\omega}), du)]} \\
&= \frac{\dot{\mathbb{E}} [L_{j-1}(X(\dot{\omega}), Y) \int_{\mathbb{R}} \varphi(u) \gamma(u, Y_j) \Lambda(X_{j-1}(\dot{\omega}), du)]}{\dot{\mathbb{E}} [L_{j-1}(X(\dot{\omega}), Y) \int_{\mathbb{R}} \gamma(u, Y_j) \Lambda(X_{j-1}(\dot{\omega}), du)]} / \dot{\mathbb{E}} [L_{j-1}(X(\dot{\omega}), Y)]
\end{aligned}$$

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$$= \frac{\mathbb{E} \left[\int_{\mathbb{R}} \varphi(u) \gamma(u, Y_j) \Lambda(X_{j-1}, du) \middle| \mathcal{F}_{j-1}^Y \right]}{\mathbb{E} \left[\int_{\mathbb{R}} \gamma(u, Y_j) \Lambda(X_{j-1}, du) \middle| \mathcal{F}_{j-1}^Y \right]}.$$

If we denote by $\pi_j(dx)$ the regular conditional distribution of X_j given $\mathcal{F}_j^Y$, then the last equation reads

$$\begin{aligned} \int_{\mathbb{R}} \varphi(x) \pi_j(dx) &= \mathbb{E} [\varphi(X_j) | \mathcal{F}_j^Y] \\ &= \frac{\mathbb{E} \left[\int_{\mathbb{R}} \varphi(u) \gamma(u, Y_j) \Lambda(X_{j-1}, du) \middle| \mathcal{F}_{j-1}^Y \right]}{\mathbb{E} \left[\int_{\mathbb{R}} \gamma(u, Y_j) \Lambda(X_{j-1}, du) \middle| \mathcal{F}_{j-1}^Y \right]} \\ &= \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(u) \gamma(u, Y_j) \Lambda(x, du) \pi_{j-1}(dx)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(u, Y_j) \Lambda(x, du) \pi_{j-1}(dx)} \\ &= \int_{\mathbb{R}} \varphi(u) \frac{\int_{\mathbb{R}} \gamma(u, Y_j) \Lambda(x, du) \pi_{j-1}(dx)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(u, Y_j) \Lambda(x, du) \pi_{j-1}(dx)} \\ &= \int_{\mathbb{R}} \varphi(x) \frac{\int_{\mathbb{R}} \gamma(x, Y_j) \Lambda(u, dx) \pi_{j-1}(du)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(x, Y_j) \Lambda(u, dx) \pi_{j-1}(du)}, \end{aligned}$$

where in the last equation we just have interchange the notation for the dummy variables in the integrals. Given the arbitrariness of φ , the result follows. Finally, we can write

$$\int_{\mathbb{R}} \varphi(x) \pi_0(dx) = \mathbb{E} [\varphi(X_0) | \mathcal{F}_0^Y] = \frac{\mathbb{E} [\varphi(X_0(\dot{\omega})) \gamma(X_0(\dot{\omega}), Y_0)]}{\mathbb{E} [\gamma(X_0(\dot{\omega}), Y_0)]} = \frac{\int_{\mathbb{R}} \varphi(x) \gamma(x, Y_0) p(dx)}{\int_{\mathbb{R}} \gamma(x, Y_0) p(dx)},$$

which gives the expression for $\pi_0(dx)$.

c (weight 10p)

We have that

$$\begin{aligned} P(X_j \in B | \mathcal{F}_{j-1}^X \vee \mathcal{F}_{j-1}^Y) &= P(\varphi(X_{j-1}) \varepsilon_j \in B | \mathcal{F}_{j-1}^X \vee \mathcal{F}_{j-1}^Y) \\ &= \mathbb{E} \left[\mathbf{1}_{\{\varphi(X_{j-1}) \varepsilon_j \in B\}} \middle| \mathcal{F}_{j-1}^X \vee \mathcal{F}_{j-1}^Y \right] \\ &= \mathbb{E} \left[\mathbf{1}_{\{\varphi(a) \varepsilon_j \in B\}} \right] \Big|_{a=X_{j-1}} = P(\varphi(a) \varepsilon_j \in B) \Big|_{a=X_{j-1}} = *. \end{aligned}$$

The distribution function of $Z := \varphi(a) \varepsilon_j$ is given by

$$P(Z \leq z) = P(\varepsilon_j \leq z/\varphi(a)) = \int_{-\infty}^{z/\varphi(a)} p(u) du = \int_{-\infty}^z p(v/\varphi(a)) \varphi(a)^{-1} dv,$$

which yields

$$* = \int_B p(u/\varphi(a)) \varphi(a)^{-1} du \Big|_{a=X_{j-1}} = \int_B p(u/\varphi(X_{j-1})) \varphi(X_{j-1})^{-1} du,$$

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and $\Lambda(x, du) = p(u/\varphi(x)) \varphi(x)^{-1} du$. On the other hand,

$$\begin{aligned} P(Y_j \in B | \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y) &= P(X_j + \xi_j \in B | \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y) \\ &= \mathbb{E} \left[\mathbf{1}_{\{X_j + \xi_j \in B\}} \middle| \mathcal{F}_j^X \vee \mathcal{F}_{j-1}^Y \right] \\ &= \mathbb{E} \left[\mathbf{1}_{\{b + \xi_j \in B\}} \right] \Big|_{b=X_j} = P(b + \xi_j \in B) \Big|_{b=X_j} = ** \end{aligned}$$

The distribution function of $W := b + \xi_j$ is given by

$$P(W \leq w) = P(\xi_j \leq w - b) = \int_{-\infty}^{w-b} q(u) du = \int_{-\infty}^w q(v - b) dv,$$

which yields

$$** = \int_B q(u - b) du \Big|_{b=X_j} = \int_B q(u - X_j) du,$$

and $\gamma(x, u) = q(u - x)$ and $\nu(du) = du$. Then, by the previous items in this problem we get

$$\pi_j(dx) = \frac{\int_{\mathbb{R}} \gamma(x, Y_j) \Lambda(u, dx) \pi_{j-1}(du)}{\int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(x, Y_j) \Lambda(u, dx) \pi_{j-1}(du)} = \frac{\int_{\mathbb{R}} q(Y_j - x) p(x/\varphi(u)) \varphi(u)^{-1} \pi_{j-1}(du)}{\int_{\mathbb{R}} \int_{\mathbb{R}} q(Y_j - x) p(x/\varphi(u)) \varphi(u)^{-1} dx \pi_{j-1}(du)} dx,$$

and

$$\pi_0(dx) = \frac{q(Y_0 - x) p_{X_0}(dx)}{\int_{\mathbb{R}} q(Y_0 - x) p_{X_0}(dx)},$$

where p_{X_0} is the density of X_0 .

Problem 3

a (weight 10p)

We will use the Kalman-Bucy filter formula in continuous time. For the given system the coefficients are

$$a_t = r, \quad b_t = 0, \quad A_t = 1, \quad B_t = m,$$

and

$$\begin{aligned} d\hat{X}_t &= a_t \hat{X}_t dt + \frac{P_t A_t}{B_t^2} (dY_t - A_t \hat{X}_t dt) = r \hat{X}_t dt + \frac{P_t}{m^2} (dY_t - \hat{X}_t dt) \\ &= \left(r - \frac{P_t}{m^2} \right) \hat{X}_t dt + \frac{P_t}{m^2} dY_t, \\ \dot{P}_t &= 2a_t P_t + b_t^2 - \frac{A_t^2}{B_t^2} P_t^2 = 2r P_t - \frac{1}{m^2} P_t^2, \end{aligned}$$

with $\hat{X}_0 = \mathbb{E}[X_0] = b$ and $P_0 = \mathbb{E}[(X_0 - b)^2] = a^2$.

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b (weight 10p)

Let $Z_t := \exp\left(-\int_0^t \left(r - \frac{P_s}{m^2}\right) ds\right)$. Since Z has continuous paths and finite variation we have that $\left[\hat{X}, Z\right]_t \equiv 0$. Moreover

$$dZ_t = -\left(r - \frac{P_t}{m^2}\right) Z_t dt.$$

Hence, by integration by parts (or product formula) we get

$$\begin{aligned} d\left(\hat{X}_t Z_t\right) &= \hat{X}_t dZ_t + Z_t d\hat{X}_t + d\left[\hat{X}, Z\right]_t \\ &= -\hat{X}_t \left(r - \frac{P_t}{m^2}\right) Z_t dt + Z_t \left\{ \left(r - \frac{P_t}{m^2}\right) \hat{X}_t + \frac{P_t}{m^2} dY_t \right\} \\ &= Z_t \frac{P_t}{m^2} dY_t. \end{aligned}$$

In integral form we obtain

$$\hat{X}_t Z_t = \hat{X}_0 Z_0 + \int_0^t Z_s \frac{P_s}{m^2} dY_s,$$

and since Z is deterministic we get

$$\hat{X}_t = b + \int_0^t Z_s Z_t^{-1} \frac{P_s}{m^2} dY_s = b + \frac{1}{m} \int_0^t \exp\left(\int_s^t \left(r - \frac{P_u}{m^2}\right) du\right) P_s dY_s.$$

Problem 4

a (weight 10p)

Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a stochastic basis satisfying the usual conditions. Let S be a complete, separable, metric space (a polish space) and let X be an $\mathbb{F}$ -adapted process taking values in S . Denote by $B(S)$ the set of Borel measurable bounded functions.

Let $A : B(S) \rightarrow B(S)$ be a linear operator and write $\mathcal{D}(A)$ for the domain of A , $\mathcal{D}(A) \subset B(S)$. Assume that $\mathbf{1} \in \mathcal{D}(A)$ and $A\mathbf{1} = 0$. Denote by $\mathcal{P}(S)$ the space of probability measures on S .

Definition (Martingale problem). Let $\pi_0 \in \mathcal{P}(S)$. We say that the process X is a solution of the martingale problem for (A, π_0) if $\pi_0 = \mathcal{L}(X_0)$ and the process M^φ defined as

$$M_t^\varphi = \varphi(X_t) - \varphi(X_0) - \int_0^t A\varphi(X_s) ds, \quad t \geq 0,$$

is an $\mathbb{F}$ -martingale for any $\varphi \in \mathcal{D}(A)$.

(Continued on page 10.)

The operator A is called the (infinitesimal) generator of the process X . The process X , called the signal, is assumed to be the solution of some martingale problem (A, π_0) .

Let W be an $\mathbb{F}$ -adapted m -dimensional Brownian motion independent of X and let Y be the process satisfying the equation

$$Y_t = \int_0^t h(X_s) ds + W_t, \quad (5)$$

where h is a Borel measurable function such that

$$P\left(\int_0^t \|h(X_s)\| ds < +\infty\right) = 1, \quad t \geq 0.$$

The process Y is called the observation process. Let $\mathcal{Y}_t := \sigma(Y_s, s \in [0, t]) \vee N$, where N are the null sets in $(\Omega, \mathcal{F}, P)$.

Definition (Filtering problem). The filtering problem consists in determining the conditional distribution π_t of the signal X at time t , given the information from observing Y up to time t . That is, for $\varphi \in B(S)$ compute

$$\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{Y}_t] = \int_S \varphi(x) \pi_t(dx).$$

The main result is the following.

Theorem. *Let S be a polish space. Then there exists a $\mathcal{P}(S)$ -valued, $\mathcal{Y}_t$ -adapted process $\pi = \{\pi_t\}_{t \geq 0}$ such that for any $\varphi \in B(S)$*

$$\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{Y}_t], \quad P\text{-a.s.}$$

Moreover, if Y satisfies (5) and h is such that

$$\mathbb{E}\left[\int_0^t \|h(X_s)\| ds\right] < +\infty,$$

and

$$P\left(\int_0^t \|\pi_s(h)\|^2 ds < +\infty\right) = 1,$$

for all $t \geq 0$, then π has a $\mathcal{Y}_t$ -adapted progressively measurable modification.

Furthermore, if X is càdlàg then π can also be chosen to be càdlàg.

Finally, the previous conditions on h are implied by

$$\mathbb{E}\left[\int_0^t \|h(X_s)\|^2 ds\right] < +\infty,$$

for all $t \geq 0$, which is usually easier to check.

(Continued on page 11.)

b (weight 10p)

Let Z be the process defined by

$$Z_t = \exp \left\{ - \int_0^t h^T(X_s) dW_s - \frac{1}{2} \int_0^t \|h(X_s)\|^2 ds \right\}, \quad t \geq 0,$$

and suppose that

$$\mathbb{E} \left[\int_0^t \|h(X_s)\|^2 ds \right] < +\infty, \quad \mathbb{E} \left[\int_0^t Z_s \|h(X_s)\|^2 ds \right] < +\infty.$$

Under these conditions we can define a probability $\tilde{P}$ on $\cup_{t \geq 0} \mathcal{F}_t$ which is equivalent to P . Under $\tilde{P}$, Y is a Brownian motion independent of X and the law of X is the same as under P . We will denote by $\tilde{\mathbb{E}}$, the expectation under $\tilde{P}$. The Kallianpur-Striebel formula, which holds under the previous hypothesis, allows to write

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)},$$

where $\rho_t(\varphi) = [\varphi(X_t) | \mathcal{Y}_t]$ is called the unnormalised conditional distribution. In what follows we will also assume that for all $t \geq 0$ we have that

$$\tilde{P} \left(\int_0^t |\rho_s(\|h\|)|^2 ds < +\infty \right) = 1.$$

We have the following result

Theorem 2 (Zakai equation). *Under the previous hypotheses ρ_t satisfies the following evolution equation*

$$\rho_t(\varphi) = \pi_0(\varphi) + \int_0^t \rho_s(A\varphi) ds + \int_0^t \rho_s(\varphi h^T) dY_s, \quad \tilde{P}\text{-a.s.},$$

for all $\varphi \in \mathcal{D}(A)$.

We also need the following technical lemma

Lemma. *Under the previous hypothesis we can write*

$$\rho_t(1) = \exp \left(\int_0^t \pi_s(h^T) dY_s - \frac{1}{2} \int_0^t \|\pi_s(h)\|^2 ds \right).$$

Now we are ready to state and proof Kushner-Stratonovich equation.

Theorem (Kushner-Stratonovich). *Under the previous hypotheses π_t satisfies the following evolution equation*

$$\pi_t(\varphi) = \pi_0(\varphi) + \int_0^t \pi_s(A\varphi) ds + \int_0^t \{ \pi_s(\varphi h^T) - \pi_s(h^T) \pi_s(\varphi) \} \{ dY_s - \pi_s(h) ds \},$$

for any $\varphi \in \mathcal{D}(A)$.

(Continued on page 12.)

Proof. From the previous lemma we get that

$$\rho_t^{-1}(1) = \exp \left(- \int_0^t \pi_s(h^T) dY_s + \frac{1}{2} \int_0^t \|\pi_s(h)\|^2 ds \right)$$

and

$$d(\rho_t^{-1}(1)) = \rho_t^{-1}(1) \left\{ -\pi_t(h^T) dY_t + \|\pi_t(h)\|^2 dt \right\}.$$

Using integration by parts, Zakai's equation for $\rho_t(\varphi)$, the previous equation and Kallianpur-Striebel formula we get

$$\begin{aligned} d\pi_t(\varphi) &= d \left(\frac{\rho_t(\varphi)}{\rho_t(1)} \right) = \rho_t^{-1}(1) d\rho_t(\varphi) + \rho_t(\varphi) d(\rho_t^{-1}(1)) + d \langle \rho_t(\varphi), \rho_t^{-1}(1) \rangle_t \\ &= \rho_t^{-1}(1) \left\{ \rho_t(A\varphi) dt + \rho_t(\varphi h^T) dY_t \right\} + \rho_t(\varphi) \rho_t^{-1}(1) \left\{ -\pi_s(h^T) dY_s + \|\pi_s(h)\|^2 ds \right\} \\ &\quad - \rho_t^{-1}(1) \rho_t(\varphi h^T) \pi_t(h) dt \\ &= \left\{ \pi_t(\varphi h^T) - \pi_t(\varphi) \pi_t(h^T) \right\} dY_t \\ &\quad + \left\{ \pi_t(A\varphi) + \pi_t(\varphi) \|\pi_t(h)\|^2 - \pi_t(h) \pi_t(\varphi h^T) \right\} dt, \end{aligned}$$

which yields the result by reordering terms. □

(Continued on page 13.)