

Theorem 2.8

Let $\mathcal{G}$ be a sub- σ -algebra of $\mathcal{F}$ and let $(S, \mathcal{S})$ be a Borel space. Any random element $X: \Omega \rightarrow S$ admits a regular conditional distribution given $\mathcal{G}$.

Proof (Sketch - Borel-Carion break)

Consider the case $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

1st Construct a regular version of $P(X \leq x | \mathcal{G})$

Define a countable family of r.v. by selecting rationals $Q_q(\omega) = P(X \leq q | \mathcal{G})(\omega)$.

For $r, q \in \mathbb{Q}$ define

$$M_{r,q} := \{\omega : Q_r < Q_q\}$$

and then, define the set where the monotonicity of the distribution function fails

$$M := \bigcup_{\substack{r > q \\ r, q \in \mathbb{Q}}} M_{r,q}$$

Due to the monotonicity property of the C.F.

and the definition of $M_{r,q}$, we get that $P(M) = 0$.

Similarly, we can define

$$N_q := \{\omega : \lim_{r \uparrow q} Q_r \neq Q_q\}$$

and $N = \bigcup_{q \in \mathbb{Q}} N_q$.

By conditional version of the monotone convergence theorem we have that $P(U_q) = 0$ and $P(U) = 0$.

Finally define

$$L_\infty := \{ \omega : \lim_{\substack{q \rightarrow \infty \\ q \in \mathbb{Q}}} Q_q \neq 1 \}$$

and again $P(L_\infty)$

$$L_{-\infty} := \{ \omega : \lim_{\substack{q \rightarrow -\infty \\ q \in \mathbb{Q}}} Q_q \neq 0 \}$$

$P(L_{-\infty})$
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0

Define

$$F(x|y) := \begin{cases} \lim_{\substack{n \uparrow x \\ n \in \mathbb{Q}}} Q_n & \text{if } \omega \notin L_\infty \cup L_{-\infty} \\ \Phi(x) & \text{otherwise.} \end{cases}$$

where $\Phi(x)$ is the c.d.f. of the normal $N(q, 1)$.
(arbitrary choice)

It follows using cond. M.C. theorem applied to the functions $f_{r_i} = 1_{(-\infty, r_i]}$, $r_i \in \mathbb{Q}$ a sequence such that $r_i \uparrow x$ that $F(x|y)$ satisfies all the properties of a distribution function and is a version of $P(X \leq x | y)$.

By a monotone class argument this distribution function can be extended to define a measure $Q(\cdot | y)$.

In the general case: $\exists \xi : S \rightarrow \mathbb{R}$ such
that ξ is $\mathcal{G}$ -measurable and ξ^{-1} is $\mathcal{B}(\mathbb{R})$ -meas.

Then $Y = \xi(X)$ is a real-valued random
variable and so it has a reg. cond. distrib.
such that for $B \in \mathcal{B}(\mathbb{R})$,

$$\hat{Q}(B|y) = P(Y \in B | y)$$

thus define

$$Q(B|y) := \hat{Q}(\xi(B)|y)$$

and since ξ^{-1} is measurable it follows that
 Q satisfies the definition of reg. cond. distrib.