Outline of the course

1. Linear Filtering in Discrete Time
   - Introduction
   - Orthogonal projection and estimation
   - Recursive orthogonal projection
   - The Kalman-Bucy filter in discrete time
   - Properties of the Kalman-Bucy filter
2. Nonlinear Filtering in Discrete Time
   - Introduction
   - The conditional expectation: a closer look
   - The Bayes formula: an abstract formulation
   - The nonlinear filter via the Bayes formula
   - The nonlinear filter via the reference measure approach
   - The linear Gaussian case: Kalman-Bucy filter revisited
3. Review on Stochastic Analysis

Introduction I

- Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing family of σ-algebras $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \subseteq \mathcal{F}$ (a filtration).
- We say that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space or a stochastic basis.
- We say that a stochastic basis satisfies the usual conditions if $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and $\mathbb{F}$ is right continous, i.e., $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$.

Definition 3.1

A stochastic process $X = \{X_t\}_{t \geq 0}$ is a family or random variables indexed by $t \in \mathbb{R}_+$.

Definition 3.2

A stochastic process X is said to be (jointly) measurable if the mapping $(t, \omega) \to X_t(\omega)$ is $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable.

Introduction II

Definition 3.3

A stochastic process X is said to be $(\mathbb{F}\text{-})\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Definition 3.4

An $\mathbb{F}$-Brownian motion (or Wiener process) $W = (W_t)_{t \geq 0}$ is a stochastic process satisfying

1. $W_0 = 0$, P-a.s,
2. W is $\mathbb{F}$-adapted,
3. $W_t - W_s$ is independent of $\mathcal{F}_s$, for all $0 \leq s \leq t$,
4. $W_t - W_s \sim N(0, t-s)$.

- The construction and fine properties of the Brownian motion are non-trivial.
- One can construct a version of the process with paths that are continous but of infinite variation, P-a.s..

Introduction III

Theorem 3.5 (Stieltjes integral)

Let $f : [0, T] \to \mathbb{R}$ be a continuous function and $g : [0, T] \to \mathbb{R}$ be a finite variation function. Let $\pi^n = \{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$ be a sequence of partitions of the interval $[0, T]$, with

$$\delta^n := \max_{0 \leq j \leq n-1} \left\{ t_{j+1}^n - t_j^n \right\} \underset{n \to \infty}{\longrightarrow} 0.$$

Then,

$$\int_0^T f(t)\, dg(t) := \lim_{n \to \infty} \sum_{j=0}^{n-1} f\left(t_j^{n,*}\right) \left(g\left(t_{j+1}^n\right) - g\left(t_j^n\right)\right),$$

exists and it is unique for any choice of $t_j^{n,*} \in \left[t_j^n, t_{j+1}^n\right]$.
It is called the Stieltjes integral of f with respect to g.

- This type of integral cannot be applied when the integrator is a Brownian motion.

The Itô integral I

Definition 3.6

A process $X = \{X_t\}_{t\in[0,T]}$ is said to belong to $\mathcal{H}^2([0,T])$ $(\mathcal{H}^2_{loc}([0,T]))$ if

1. X is measurable,
2. X is $\mathbb{F}$-adapted,
3. $\mathbb{E}\left[\int_0^T X_t^2 dt\right] < +\infty, \qquad \left(P\left(\int_0^T X_t^2 dt < +\infty\right) = 1\right).$

The Itô integral II

Definition 3.7

A process X is $\mathcal{H}^2([0,T])$-simple if it belongs to $\mathcal{H}^2([0,T])$ and has the form

$$X_t = \sum_{j=0}^{n-1} \xi_j \mathbf{1}_{[t_j, t_{j+1})}(t)$$

for some partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0,T]$ and random variables $\{\xi_j\}_{j=0,\ldots,n}$.

Definition 3.8

The Itô integral for a simple process X is defined by

$$I_T(X) := \int_0^T X_t dW_t := \sum_{j=0}^{n-1} \xi_j \left(W_{t_{j+1}} - W_{t_j}\right).$$

The Itô integral III

Theorem 3.9

Let $X \in \mathcal{H}^2([0,T])$ and let X^k be a sequence of $\mathcal{H}^2([0,T])$-simple processes such that

$$\int_0^T \mathbb{E}\left[\left|X_t - X_t^k\right|^2\right] dt \underset{k\to\infty}{\longrightarrow} 0.$$

Such a sequence always exists and the limit

$$\int_0^T X_t dW_t := L^2\text{-}\lim_{k\to\infty} \int_0^T X_t^k dW_t,$$

always exists and it is independent of the sequence. We call this limit the Itô integral of X with respect to W.

Properties of the Itô integral

Proposition 3.10

Let X and Y be two processes in $\mathcal{H}^2([0,T])$. Then

1. $\int_0^T X_t dW_t = \int_0^S X_t dW_t + \int_S^T X_t dW_t$, $S \leq T$,
2. $\int_0^T (aX_t + bY_t)\, dW_t = a\int_0^T X_t dW_t + b\int_0^T Y_t dW_t$, $a,b \in \mathbb{R}$,
3. $\mathbb{E}\left[\int_0^T X_t dW_t\right] = 0$,
4. $\mathbb{E}\left[\int_0^T X_t dW_t \int_0^S Y_t dW_t\right] = \int_0^{S\wedge T} \mathbb{E}\left[X_t Y_t\right] dt$. *In particular*

$$\mathbb{E}\left[\left(\int_0^T X_t dW_t\right)^2\right] = \int_0^T \mathbb{E}\left[X_t^2\right] dt,$$

5. $\int_0^t X_s dW_s, t \in [0,T]$ *is an $\mathbb{F}$-adapted process,*
6. $\int_0^t X_s dW_s, t \in [0,T]$ *admits a continuous version and it is a martingale.*

Remark 3.11

If X only belongs to $\mathcal{H}^2_{loc}([0,T])$, properties 3 and 4 may fail and the integral (as a process) is only a local martingale.

Stochastic differential equations I

- Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis and W be a $\mathbb{F}$-Brownian motion.
- Let $a(t,x)$ and $b(t,x)$ be a pair of functionals on $C([0,T],\mathbb{R})$, which are non-anticipating in the sense that for each $t \in [0,T]$ fixed

$$x_1(s) \equiv x_2(s), \quad s \leq t \Rightarrow a(t,x_1) = a(t,x_2), \quad b(t,x_1) = b(t,x_2).$$

Equivalently, for each $t \in [0,T]$ fixed, $a(t,x)$ and $b(t,x)$ are $\mathcal{B}(C([0,t],\mathbb{R}))$-measurable.

Stochastic differential equations II

Definition 3.12

We shall say that the (P-a.s continuous) stochastic process $X = \{X_t\}_{t\in[0,T]}$ is a strong solution (or simply a solution) of the differential equation

$$dX_t = a(t, X)\, dt + b(t, X)\, dW_t, \tag{3.1}$$

with the $\mathcal{F}_0$-measurable initial condition $X_0 = \eta$ if for each

- X is $\mathbb{F}$-adapted,
- the following integrability condition is satisfied

$$P\left(\int_0^T |a(t, X)|\, dt < \infty\right) = P\left(\int_0^T |b(t, X)|^2\, dt < \infty\right) = 1,$$

- and for all $0 \leq t \leq T$

$$X_t = \eta + \int_0^t a(s, X)\, ds + \int_0^t b(s, X)\, dW_s, \quad P\text{-a.s.}$$

Stochastic differential equations III

Definition 3.13

One says that the SDE given by (3.1) has a unique strong solution if for any two of its strong solutions X and $\tilde{X}$ we have that

$$P\left(\sup_{0\leq t\leq T}\left|X_t-\tilde{X}_t\right|>0\right)=0.$$

Stochastic differential equations IV

Theorem 3.14

Let the nonanticipative functionals $a(t,x)$ and $b(t,x)$ satisfy for all $t \in [0,T]$ the following Lipschitz condition

$$|a(t,x)-a(t,y)|^2+|b(t,x)-b(t,y)|^2 \leq L_1 \int_0^t |x(s)-y(s)|^2 dK_s + L_2 |x(t)-y(t)|^2,$$

and the growth condition

$$a^2(t,x)+b^2(t,x) \leq L_1 \int_0^t \left(1+x^2(s)\right) dK_s + L_2 \left(1+x^2(t)\right),$$

where L_1, L_2 are constants, K_s is a nondecreasing right continuous function, $0 \leq K_s \leq 1$,$x, y \in C([0,T],\mathbb{R})$.
Let η be a $\mathcal{F}_0$- measurable random variable (finite P-a.s). Then,

1. *the equation (3.1) has a unique strong solution,*
2. *if $\mathbb{E}\left[|\eta|^{2m}\right] < \infty$, $m \geq 1$, then there exists a constant c_m such that*

$$\mathbb{E}\left[|X_t|^{2m}\right] \leq \left(1+\mathbb{E}\left[|\eta|^{2m}\right]\right) e^{c_m t} - 1.$$

Stochastic differential equations V

Corollary 3.15

Consider the SDE

$$dX_t = a(t, X_t)\, dt + b(t, X_t)\, dW_t, \tag{3.2}$$

where the functions $a(t,z)$ and $b(t,z)$, $0 \leq t \leq T$, $z \in \mathbb{R}$ satisfy the Lipschitz condition

$$|a(t,z) - a(t,\tilde{z})|^2 + |b(t,z) - b(t,\tilde{z})|^2 \leq L_2 |z - \tilde{z}|,$$

and

$$a^2(t,z) + b^2(t,z) \leq L\left(1 + z^2\right).$$

Then, equation (3.2) with the initial condition $X_0 = \eta$, finite P-a.s., has a unique strong solution.

Main results on Itô integration I

Theorem 3.16 (**Itô formula**)

Let X have the Itô differential

$$dX_t = a_t dt + b_t dW_t, \qquad t \in [0, T],$$

where a_t and b_t are n dimensional vector and $n \times m$ matrix of $\mathcal{H}^2([0,T])$-stochastic processes and W is an m-dimensional Wiener process. Assume $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is $C^{1,2}([0,T] \times \mathbb{R}^n)$. Then

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\, dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(t, X_t)\, dX_t^i + \frac{1}{2} \sum_{i,j=1}^{n} c_i^j(t) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)\, dt,$$

where $c = \left(c_i^j(t)\right)_{t \in [0,T], i,j=1,\ldots,n}$ is called the diffusion matrix of X and it is defined by $c = bb^T$.

Main results on Itô integration II

Theorem 3.17 (**Girsanov's theorem**)

Let $\beta = \{\beta_t\}_{t\in[0,T]}$ *be an* $\mathbb{F}$*-adapted process, defined on* $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *and satisfying*

$$P\left(\int_0^T \beta_t^2 dt < \infty\right) = 1,$$

and let

$$Z_t := \exp\left(\int_0^t \beta_s dW_s - \frac{1}{2}\int_0^t \beta_s^2 ds\right).$$

Assume that $\mathbb{E}[Z_T] = 1$ *and define the probability measure* $\tilde{P}$ *by* $\frac{d\tilde{P}}{dP} = Z_T$*. Then,*

$$\tilde{W}_t := W_t - \int_0^t \beta_s ds, \qquad t \in [0, T],$$

is a Wiener process with respect to $\mathbb{F}$ *under* $\tilde{P}$*.*

Main results on Itô integration III

Remark 3.18

To verify $\mathbb{E}[Z_T] = 1$ is not easy in general. One of the simplest criteria to verify to verify it is Novikov's condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \beta_t^2 dt\right)\right] < +\infty.$$

Theorem 3.19 (**Itô representation theorem**)

Let ξ be a square integrable, $\mathcal{F}_T^W$-measurable random variable. Then, there exists a unique $\mathcal{H}^2([0,T])$-process f such that

$$\xi = \mathbb{E}[\xi] + \int_0^T f(t)\, dW_t.$$

Main results on Itô integration IV

Theorem 3.20 (The martingale representation theorem)

Let X be a square integrable, $\mathbb{F}^W$-martingale. Then, there exists a unique $\mathcal{H}^2([0,T])$-process g such that

$$X_t = \mathbb{E}[X_T] + \int_0^t g(s)\, dW_s, \qquad t \in [0,T].$$

More results on martingales I

Definition 3.21

A family of random variables $\{X_\alpha\}_{\alpha\in\Lambda}$ is uniformly integrable if

$$\lim_{K\to+\infty}\sup_{\alpha\in\Lambda}\mathbb{E}\left[|X_\alpha|\,\mathbf{1}_{\{|X_\alpha|\geq K\}}\right]=0.$$

Definition 3.22

The predictable (optional) σ-algebra $\mathcal{P}$ ($\mathcal{O}$) is the σ-algebra on $[0,\infty)\times\Omega$ generated by the left continuous (right continuous) adapted processes.

More results on martingales II

Definition 3.23

Let $A = \{A_t\}_{t \geq 0}$ be a right continuous, adapted process.

- A is called increasing if $A_0 = 0$, P-a.s, has nondecreasing paths P-a.s, and $\mathbb{E}[A_t] < +\infty$, $t \geq 0$,
- A is called integrable if $\mathbb{E}[A_\infty] < +\infty$, where $A_\infty = \lim_{t \to \infty} A_t$ (whenever it does exist).

More results on martingales III

Definition 3.24

Let $X = \{X_t\}_{t \geq 0}$ be a right continuous, adapted process

- X is said to belong to the class D if the family of random variables $\{X_\tau\}_{\tau \in \mathcal{T}}$ is uniformly integrable, where $\mathcal{T}$ is the set of all finite stopping times.
- X is said to belong to the class DL if for all $a > 0$ the family of random variables $\{X_\tau\}_{\tau \in \mathcal{T}_a}$ is uniformly integrable, where $\mathcal{T}_a$ is the set of all stopping times bounded by a.

Theorem 3.25

1. *Any right continuous martingale belongs to the class DL.*
2. *Any right continuous uniformly integrable martingale belongs to the class D.*
3. *Any right continous nonnegative submartingale belongs the the class DL.*

More results on martingales IV

Theorem 3.26 (**Doob-Meyer decomposition**)

Let X be a right continuous submartingale of class DL. Then X has the following decomposition

$$X = A + M,$$

where M is a right continuous martingale and A is a predictable increasing process. This decomposition is unique up to indistinguishability. If X belongs to the class D, then M is uniformly integrable and A is integrable.

Proposition 3.27

If X is a continuous nonnegative submartingale, the processes M and A in the Doob-Meyer decomposition can be taken to be continuous.

More results on martingales V

Definition 3.28

A right continuous martingale X is called square integrableif $\mathbb{E}\left[X_t^2\right] < +\infty$ for all $t \geq 0$, and denote by $\mathcal{M}_0^2$ the class of all square integrable martingales starting at 0 and by $\mathcal{M}_0^{2,c} \subseteq \mathcal{M}_0^2$, the ones with continuous paths.

- For $X \in \mathcal{M}_0^2$, one can check that X^2 is a submartingale. Since it is nonnegative, it is of class DL and the Doob-Meyer decomposition applies.

Definition 3.29

For $X \in \mathcal{M}_0^2$, the process A in the Doob-Meyer decomposition is called the predictable quadratic variation process and it is denoted by $\langle X \rangle$.
So $\langle X \rangle$ is the unique predictable, increasing process that makes $X^2 - \langle X \rangle$ a martingale.

More results on martingales VI

Remark 3.30

If X is continuous $\langle X \rangle$ is also continuous.

Proposition 3.31

If $X, Y \in \mathcal{M}_0^2$, the process $\langle X, Y \rangle$ defined by

$$\langle X, Y \rangle = \frac{1}{4} \left(\langle X + Y \rangle - \langle X - Y \rangle \right),$$

is the unique predictable process with finite variation paths (difference of two increasing processes) such that

$$XY - \langle X, Y \rangle,$$

is a martingale. $\langle X, Y \rangle$ is called the predictable quadratic covariation of X and Y.

More results on martingales VII

Definition 3.32

Let $X \in \mathcal{M}_0^2$. We define for each $t \geq 0$ the function

$$\|X\|_t = \left(\mathbb{E}\left[X_t^2\right]\right)^{1/2},$$

and

$$\|X\|_\infty = \sup_{t \geq 0} \|X\|_t .$$

We say that $X \in \mathcal{M}_0^2$ is L^2-bounded if and only if $\|X\|_\infty < +\infty$.
For all $X \in \mathcal{M}_0^2$, we define

$$\|X\| = \sum_{n=1}^{\infty} 2^{-n} \left(\|X\|_n \wedge 1\right),$$

and $d(X, Y) = \|X - Y\|$, for all $X, Y \in \mathcal{M}_0^2$.

More results on martingales VIII

Remark 3.33

If we identify processes that are indistinguishable, then $\left(\mathcal{M}_0^2, d\right)$ becomes a metric space and, on the subclass of L^2-bounded martingales, the function $\|\cdot\|_\infty$ becomes a norm.

Proposition 3.34

The metric space $\left(\mathcal{M}_0^2, d\right)$ is complete and $\mathcal{M}_0^{2,c}$ is a closed subspace of $\mathcal{M}_0^2$ and thus complete as well.

Definition 3.35

A property π is said to holdlocally for a stochastic process X if there exists a sequence of stopping times $\{\tau_n\}_{n\geq 1}$ increasing to ∞ a.s. such that $X_{t\wedge\tau_n}\mathbf{1}_{\{\tau_n>0\}}$ has the property π for all $n \geq 1$.

- The indicator $\mathbf{1}_{\{\tau_n>0\}}$ is for allowing X_0 to not have the property π.

More results on martingales IX

Definition 3.36

An adapted right continuous with left limits process X is a local martingale if X is locally in the class of uniformly integrable martingales.

Definition 3.37

A semimartingale is a process X of the form $X = X_0 + M + A$, where X_0 is $\mathcal{F}_0$-measurable random variable, M is a local martingale and A is a finite variation process.

More results on martingales X

Definition 3.38

Let H be a locally bounded predictable process and X be a semimartingale. Then

$$\int_0^t H_s dX_s = \int_0^t H_s dM_s + \int_0^t H_s dA_s, \qquad 0 \leq t \leq T.$$

If X is locally L^2-bounded the class of integrands can be extended to

$$\mathcal{L}_M^2 := \left\{ H \quad \text{predictable} : \mathbb{E}\left[\int_0^T H_s^2 d\langle M \rangle_s \right] < +\infty \right\}.$$

More results on martingales XI

Remark 3.39

The space of semimartingales is pleasant because it is stable under

- stopping,
- localization,
- random time changes,
- absolutely continuous change of measure,
- changes of filtration (+ some conditions)
- smooth transformations (Itô formula),

and it is the largest class with respect to one can integrate all bounded predictable processes and the integral is a semimartingale.

More results on martingales XII

Remark 3.40

Let $X = X_0 + M + A$ be a semimartingale

- X is continuous if and only if M and A are continuous. Such decomposition is unique.
- If X is continuous then $\langle X \rangle = \langle M \rangle$.

Lemma 3.41 (**Itô formula**)

Let X be and $\mathbb{R}^d$-valued continous semimartingale and $f \in C^{1,2}\left([0,T] \times \mathbb{R}^d\right)$. Then

$$f(t, X_t) = f(X_0) + \int_0^t \frac{\partial}{\partial t} f(s, X_s)\, ds + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f(s, X_s)\, dX_s^i$$
$$+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s)\, d\left\langle X^i, X^j \right\rangle_s .$$

More results on martingales XIII

Lemma 3.42 (**Integration by parts**)

Let X and Y be two continuous semimartingales. Then

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t .$$

Theorem 3.43 (**Lévy's characterization theorem**)

Let $B = \left(B^1, ..., B^d\right)$ be a d-dimensional continuous local martingale with respect to a filtration $\mathbb{F}$, starting at 0.
Then,

$$B \text{ is standard } \mathbb{F}\text{-Brownian motion} \iff \left\langle B^i, B^j \right\rangle_t = \delta_{ij} t, \qquad \forall i, j \in \{1, ..., d\}$$

More results on martingales XIV

Theorem 3.44 (Burkholder-Davis-Gundy inequality)

Let M be a continuous martingale and let $M_t^* = \sup_{s \leq t} |M_s|$. Then, for all $p \geq 1$, there exists constants c_p and C_p (not depending on M) such that

$$c_p \mathbb{E}\left[\langle M \rangle_t^{p/2}\right] \leq \mathbb{E}\left[M_t^*\right] \leq C_p \mathbb{E}\left[\langle M \rangle_t^{p/2}\right].$$