Complex Analysis
## Contents

Chapter 1. Fundamental Properties of Holomorphic Functions 5  
1. Basic definitions 5  
2. Integration and Integral formulas 6  
3. Some consequences of the integral formulas 8  

Chapter 2. Runge’s Theorem 13  
1. Partitions of unity 13  
2. Smeared out Cauchy Integral Formula 13  
3. Runge’s Theorem 13  

Chapter 3. Applications of Runge’s Theorem 19  
1. The $\overline{\partial}$-equation 19  
2. The theorems of Mittag-Leffler and Weierstrass 20  

Chapter 4. Piccard’s Theorem 23  

Chapter 5. Riemann mapping theorem 25  

Chapter 6. Some exercises fro Narasimhan/Nievergelt 27  

Chapter 7. From Forster’s Book, Lectures on Riemann Surfaces 29  
1. DeRham-Hodge Theorem 29
CHAPTER 1

Fundamental Properties of Holomorphic Functions

1. Basic definitions

Definition 1.1. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f = u + iv \in C^1(\Omega)$, where $u, v$ are real functions. We say that $f$ is holomorphic on $\Omega$ if for any point $a \in \Omega$ we have that

\begin{equation}
\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \quad \text{and} \quad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a).
\end{equation}

The equations (1.1) are called the Cauchy-Riemann equations. We define the following differential operators:

Definition 1.2. Let $\Omega \subset \mathbb{C}$ be an open set, and let $f = u + iv$ be differentiable at every point of $\Omega$. We set

\begin{equation}
\frac{\partial f}{\partial z}(a) := \frac{1}{2} (\frac{\partial f}{\partial x}(a) - i \frac{\partial f}{\partial y}(a)) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}}(a) := \frac{1}{2} (\frac{\partial f}{\partial x}(a) + i \frac{\partial f}{\partial y}(a)).
\end{equation}

We see that the condition that $\frac{\partial f}{\partial \bar{z}}(a) = 0$ is satisfied is equivalent to the conditions (1.1) being satisfied.

Lemma 1.3. Let $f \in C^1(\Omega)$ and let $a \in \Omega$. Then

\begin{equation}
f(z) = f(a) + \frac{\partial f}{\partial z}(a) \cdot (z - a) + \frac{\partial f}{\partial \bar{z}}(a) \cdot \overline{(z - a)} + O(|z|^2).
\end{equation}

Proof. This follows from Taylor’s Theorem for maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ writing it on complex form.

Using this it is not hard to see that a $C^1$-smooth function $f$ on $\Omega$ is holomorphic if and only if the limit

\begin{equation}
\lim_{\delta \to 0} \frac{f(a + \delta) - f(a)}{\delta}
\end{equation}

exists for all $a \in \Omega$.

If a function $f$ is holomorphic on an open set $\Omega$ the expression $\frac{\partial f}{\partial z}(a)$ is called the (complex) derivative of $f$ at $a \in \Omega$, and we denote this also by $f'(a)$. We denote the set of holomorphic functions on $\Omega$ by $O(\Omega)$, and we note that $O(\Omega)$ is an algebra, i.e., if $f, g \in O(\Omega)$ then $f + g, f - g, f \cdot g$ are holomorphic on $\Omega$. If $f$ is non-zero on $\Omega$ then $1/f$ is holomorphic on $\Omega$. Moreover, the usual rules of differentiation hold: $(f + g)' = f' + g', (f \cdot g)' = f' \cdot g + f \cdot g'$.
1. FUNDAMENTAL PROPERTIES OF HOLOMORPHIC FUNCTIONS

If \( f \cdot g' + f' \cdot g, (1/g)' = -g'/g^2 \). If \( f \in \mathcal{O}(\Omega_1) \) and \( g \in \mathcal{O}(\Omega_2) \) and \( f(a) \in \Omega_2 \), then the composition \( g \circ f \) is holomorphic at \( a \), and \( (g \circ f)' = g'(f(a)) \cdot f'(a) \).

**Example 1.4.** A polynomial \( P(z) = a_n \cdot z^n + \cdots + a_1 \cdot z + a_0 \) is holomorphic.

2. Integration and Integral formulas

We will start by proving the fundamental result that if \( f \in C^1(D) \) and \( f \) is holomorphic on \( D \) then

\[
(2.1) \quad f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - a} \, dz,
\]

for all \( a \in D \). Actually we will prove the more general result that if \( \Omega \) is a bounded \( C^1 \)-smooth domain and if \( f \in C^1(\Omega) \), then

\[
(2.2) \quad f(a) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{z - a} \, dz - \frac{1}{\pi} \int \int_{\Omega} \frac{\partial f(z)}{z - a} \, dxdy.
\]

**Definition 2.1.** Let \( f, g \in C^k(\Omega) \). We call the expression \( \omega(z) = f(z) \cdot dx + g(z) \cdot dy \) a differentiable 1-form of class \( C^k \). We denote the vector space of differentiable 1-forms on \( \Omega \) by \( E^1(\Omega) \).

**Definition 2.2.** We set

\[
(2.3) \quad dz := dx + i \cdot dy \text{ and } d\overline{z} := dx - i \cdot dy.
\]

**Definition 2.3.** Let \( \omega \in E^1(\Omega) \) be a continuous 1-form, and let \( \gamma : [0, 1] \rightarrow \Omega \) be a \( C^1 \)-smooth map. We set

\[
(2.4) \quad \int_{\gamma} \omega := \int_{0}^{1} f(\gamma(t)) \cdot \gamma'_1(t) + g(\gamma(t)) \cdot \gamma'_2(t) \, dt.
\]

**Proposition 2.4.** Let \( \gamma, \sigma : [0, 1] \rightarrow \Omega \) be two parametrizations of the same curve, i.e., \( \gamma = \sigma \circ \phi \) where \( \phi : [0, 1] \rightarrow [0, 1] \) is a strictly increasing \( C^1 \)-smooth function, \( \phi(0) = 0, \phi(1) = 1 \). Then for \( \omega \in E^1(\Omega) \) continuous, we have that

\[
(2.5) \quad \int_{\gamma} \omega = \int_{\sigma} \omega.
\]

**Proof.** We have that

\[
\int_{0}^{1} f(\gamma(t)) \cdot \gamma'_1(t) \, dt = \int_{0}^{1} f(\sigma \circ \phi(t)) \cdot (\sigma_1 \circ \phi)'(t) \, dt
\]

\[
= \int_{0}^{1} f(\sigma \circ \phi(t)) \cdot \sigma'_1(\phi(t)) \cdot \phi'(t) \, dt
\]

\[
= \int_{0}^{1} f(\sigma(t)) \cdot \sigma'_1(t) \, dt,
\]

where the last equation follows from the change of variables formula in one variable, and the same holds if you exchange \( f \) by \( g \) and \( \gamma_1, \sigma_1 \) by \( \gamma_2, \sigma_2 \). \( \square \)
This allows us to define integration of 1-forms on oriented curves in \( \mathbb{C} \), and furthermore on the boundaries of (piecewise) \( C^1 \)-smooth domains in \( \mathbb{C} \), as long as we orient the boundary components. The following theorem is fundamental, and is the basic ingredient to prove (2.2):

**Theorem 2.5. (Stokes)** Let \( \Omega \) be a bounded (piecewise) \( C^1 \)-smooth domain in \( \mathbb{C} \) and let \( \omega = f \, dx + g \, dy \in \mathcal{E}^1(\Omega) \) be of class \( C^1 \). Then

\[
\int_{\partial \Omega} \omega = \int_{\Omega} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy. 
\]

**Proof.** We give a complete proof only when \( \Omega \) is the square \([0, 1] \times [0, 1] \).

\[
\int_{\Omega} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_0^1 \left( \int_0^1 \frac{\partial g}{\partial x}(x,y) \, dx \right) dy - \int_0^1 \left( \int_0^1 \frac{\partial f}{\partial y}(x,y) \, dy \right) dx
\]

\[
= \int_0^1 g(1, y) - g(0, y) \, dy - \int_0^1 f(x, 1) - f(x, 0) \, dx
\]

\[
= \int_{\partial \Omega} \omega. 
\]

\[\square\]

Note that if \( \omega \) is a 1-form \( \omega(z) = f(z) \cdot dz \), then (2.6) reads

\[
\int_{\partial \Omega} \omega = 2i \cdot \int \int_{\Omega} \frac{\partial f}{\partial z}(z) \cdot dxdy
\]

In particular we get the following result:

**Proposition 2.6.** Let \( \Omega \subset \mathbb{C} \) be a bounded \( C^1 \)-smooth domain and let \( \omega \in \mathcal{E}^1(\Omega) \) be \( C^1 \)-smooth holomorphic on \( \Omega \), i.e., \( \omega(z) = f(z) \cdot dz \), \( f \in \mathcal{O}(\Omega) \). Then

\[
\int_{\partial \Omega} \omega = 0.
\]

**Theorem 2.7. (Generalized Cauchy Integral Formula)** Let \( \Omega \subset \mathbb{C} \) be a bounded \( C^1 \)-smooth domain and let \( f \in \mathcal{C}^1(\Omega) \). Then for each \( a \in \Omega \) we have that

\[
f(a) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z)}{z-a} \, dz - \frac{1}{\pi} \int \int_{\Omega} \frac{\partial f}{\partial z}(z) \cdot \frac{1}{z-a} dxdy.
\]

**Proof.** We prove first that the last integral is well defined:

**Lemma 2.8.** \( \int \int_{D^*} \frac{1}{|z|} = 2\pi. \)

**Proof.** \( \int \int_{D^*} \frac{1}{|z|} = \int_0^1 \int_0^{2\pi} \frac{1}{r \alpha} \cdot r dt d\theta = 2\pi. \)

Now for \( \epsilon \) small enough such that \( D_\epsilon(a) \subset \subset \Omega \) we set \( \Omega_\epsilon = \Omega \setminus D_\epsilon(a) \). Stoke's Theorem gives us that

\[
\frac{1}{2\pi i} \int_{\partial \Omega_\epsilon} \frac{f(z)}{z-a} \, dz = \frac{1}{\pi} \int \int_{\Omega_\epsilon} \frac{\partial f}{\partial z}(z) \cdot \frac{1}{z-a} dxdy,
\]
and furthermore
\begin{equation}
\int_{b\Omega} \frac{f(z)}{z-a} \, dz = \int_{b\Omega} \frac{f(z)}{z-a} \, dz - \int_{b\partial \epsilon(a)} \frac{f(z)}{z-a} \, dz,
\end{equation}
and we see that the last rightmost integral approaches \(2\pi i f(a)\) uniformly as \(\epsilon \to 0\). \(\square\)

### 3. Some consequences of the integral formulas

**Proposition 3.1.** Let \(\Omega \subset \mathbb{C}\) be a bounded \(C^1\)-smooth domain and let \(f \in C(b\Omega)\). Then the function
\begin{equation}
\tilde{f}(\zeta) = \frac{1}{2\pi i} \int_{b\Omega} \frac{f(z)}{z-\zeta} \, dz
\end{equation}
is holomorphic on \(\Omega\). Moreover, \(\tilde{f}\) is \(C^\infty\)-smooth, \(\tilde{f}'\) is holomorphic, and we have that
\begin{equation}
\tilde{f}^{(k)}(\zeta) = \frac{k!}{2\pi i} \int_{b\Omega} \frac{f(z)}{(z-\zeta)^{k+1}} \, dz
\end{equation}

**Proof.** This follows by differentiating under the integral sign. \(\square\)

**Proposition 3.2.** Let \(f_j \in \mathcal{O}(\mathbb{D}) \cap C(\overline{\mathbb{D}})\) for \(j \in \mathbb{N}\), and assume that \(f_j \to f\) uniformly on \(\overline{\mathbb{D}}\) as \(j \to \infty\). Then \(f \in \mathcal{O}(\mathbb{D})\), and \(f^{(k)} \to f^{(k)}\) uniformly on compact subsets of \(\mathbb{D}\) as \(j \to \infty\).

**Proof.** Note that \(f\) is given by an integral formula as in the previous proposition. \(\square\)

**Definition 3.3.** We say that a function \(f\) on \(D_r(0)\) is **analytic** if \(f(z) = \sum_{j=0}^{\infty} c_j \cdot z^j\) for all \(z \in D_r\).

**Proposition 3.4.** If \(f\) is analytic on \(D_r\), then \(f \in \mathcal{O}(D_r)\).

**Proof.** Fix a \(0 < t < s < r\), and note that there exists \(M > 0\) such that \(|c_j \cdot s^j| < M\) for all \(j \in \mathbb{N}\). Then for all \(z \in D_t\) we have that
\begin{equation}
|\sum_{j=N}^{\infty} c_j \cdot z^j| \leq \sum_{j=N}^{\infty} |c_j \cdot s^j| \cdot \left(\frac{t}{s}\right)^j \leq M \sum_{j=N}^{\infty} \left(\frac{t}{s}\right)^j.
\end{equation}
By the convergence of geometric series, this shows that \(f\) is the limit of a sequence of polynomials on \(D_t\) for all \(t < r\), hence \(f\) is holomorphic on \(D_r\) by Proposition 3.4 \(\square\)

**Proposition 3.5.** (Cauchy Estimates) Let \(f \in \mathcal{O}(D_r) \cap C(\overline{D}_r)\). Then
\begin{equation}
|f^{(k)}(0)| \leq \frac{k! \cdot \|f\|_{D_r}}{r^k}.
\end{equation}
PROOF. By (3.2) we have that
\[ |f^{(k)}(0)| \leq \frac{k!}{2\pi} \int_{\partial D_r} \frac{f(z)}{z^{k+1}} |dz| \]
\[ = \frac{k!}{2\pi} \int_0^{2\pi} f(re^{it}) \left( \frac{dz}{re^{it}} \right) |dz| \]
\[ \leq \frac{k! \cdot \|f\|_{\partial D_r}}{r^k}. \]
\[ \square \]

**Corollary 3.6.** *(Simple Maximum principle for a disk)* Let \( f \in \mathcal{O}(D_r) \cap \mathcal{C}(\overline{D_r}) \). Then \( |f(0)| \leq \|f\|_{\partial D_r} \).

**Theorem 3.7.** *(Montel)* Let \( \Omega \subset \mathbb{C} \) be an open set, and \( \mathcal{F} \) be a family of holomorphic functions on \( \Omega \) with the property that for each compact set \( K \subset \Omega \) there exists a constant \( C_K > 0 \) such that \( \|f\|_K \leq C_K \) for all \( f \in \mathcal{F} \). Then for any sequence \( \{f_j\}_{j \in \mathbb{N}} \subset \mathcal{F} \) there exists a subsequence \( \{f_{n(j)}\} \) such that \( f_{n(j)} \to f \in \mathcal{O}(\Omega) \) uniformly on compact subsets of \( \Omega \).

**Proof.** Let \( A \subset \Omega \) be a dense sequence of points, and let \( \{f_j\} \subset \mathcal{F} \) be a sequence such that \( f_j(a) \to \tilde{a} \in \mathcal{C} \) for all \( a \in A \). We claim that the sequence \( \{f_j\} \) converges to a holomorphic function \( f \) uniformly on compact subsets of \( \Omega \). Choose an exhaustion of \( \Omega \) by compact sets \( K_j \subset K_{j+1} \). For any \( j \) we have that \( \|f_j\|_{K_j} \leq M_j \) for all \( i \). By the Cauchy estimates there is a constant \( N_j \) such that \( \|f_j\|_{K_j} < N_j \) for all \( i \).

Now we fix \( K_j \) and show that \( \{f_i\}_{K_j} \) is a Cauchy sequence. Note that by the Mean Value Theorem we have for \( z, z' \in K_{j+1} \) that \( |f_i(z) - f_i(z')| \leq N_{j+1} |z - z'| \). Given any \( \epsilon > 0 \) we may choose a finite subset \( A \subset K_{j+1} \) of \( A \) such that for any \( z \in K_j \), there exists an \( a \in A \) with \( |z - a| < \frac{\epsilon}{4N_{j+1}} \). Furthermore, since \( \{f_i\}_{A} \) is Cauchy, we may find \( N \in \mathbb{N} \) such that \( |f_i(a) - f_n(a)| < \frac{\epsilon}{2} \) for all \( m, n \geq N \). So given any \( z \in K_j \) we may pick \( a \in A \) to see that

\[ |f_i(z) - f_m(z)| \leq |f_i(z) - f_i(a)| + |f_i(a) - f_m(a)| + |f_m(a) - f_m(z)| \]
\[ \leq 2N_{j+2} |z - a| + \epsilon/2 < \epsilon, \]
for all \( l, m \geq N \), hence \( \{f_i\}_{K_j} \) is a Cauchy sequence. \( \square \)

**Theorem 3.8.** Let \( f \in \mathcal{O}(D_r) \). Then we have that
\[ f(\zeta) = \sum_{j=0}^{\infty} c_j \cdot \zeta^j, \]
where
\[ c_j = \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(z)}{z^{j+1}} |dz|. \]
\textbf{Proof.} Note that \( \frac{1}{z-\zeta} = \frac{1}{z(1-\zeta/z)} = 1/z\sum_{j=0}^{\infty} (\zeta/z)^j \) as long as \(|\zeta| < |z|\), and plug this into Cauchy’s Integral Formula.

\textbf{Proposition 3.9. (Identity principle)} Let \( f \in \mathcal{O}(\Omega) \). If \( Z(f) = \{ z \in \Omega : f(z) = 0 \} \) has non-empty interior, then \( f \equiv 0 \) on \( \Omega \).

\textbf{Proof.} For each \( a \in \Omega \) we have that \( f(z) = \sum_{j=0}^{\infty} c_j(a)(z-a)^j \) on a small enough disk centered at \( a \). By (3.6) we see that \( c_j(a) \) is continuous in \( a \) for all \( j \). So the set of points \( \{ a \in \Omega : c_j(a) = 0 \text{ for all } j \in \mathbb{N} \} \) is non-empty, open and closed in \( \Omega \).

\textbf{Proposition 3.10.} Let \( f \in \mathcal{O}(\Omega) \). Then \( Z(f) \) is discrete unless \( f \) is constantly equal to zero.

\textbf{Proof.} We assume that \( f \) is not constant. Near a point \( a \in \Omega \) with \( f(0) = 0 \) we have that \( f(z) = \sum_{j=k}^{\infty} c_j(z-a)^j \), \( k \geq 1 \), \( c_k \neq 0 \), so we can write \( f(z) = (z-a)^k(c_k + \sum_{j=1}^{\infty} c_{k+j}(z-a)^j) \).

\textbf{Theorem 3.11. (Open Mapping Theorem)} Let \( f \in \mathcal{O}(\mathbb{D}) \) be non-constant. Then \( f(\mathbb{D}) \) is an open set.

\textbf{Proof.} Assume that \( f(0) = 0 \) but that there are points \( a_j \to 0 \) such that \( f(z) \neq a_j \) for all \( j \in \mathbb{N} \). Set \( g_j(z) := \frac{1}{f(z) - a_j} \). Choose \( r > 0 \) such that \( f(z) \neq 0 \) for all \(|z| = r\). Then \(|g_j|\) is uniformly bounded on \( bD_r \) but \( g(0) \to \infty \) as \( j \to 0 \) which contradicts the simple maximum principle for a disk.

\textbf{Corollary 3.12. (Maximum principle)} Let \( \Omega \subset \mathbb{C} \) be a domain, and let \( f \in \mathcal{O}(\Omega) \). If \(|f(a)| = \sup_{z \in \Omega} |f(z)| \), \( a \in \Omega \), then \( f \) is constant.

\textbf{Proposition 3.13.} Let \( \Omega \subset \mathbb{C} \) be a domain. Let \( f_j \in \mathcal{O}^*(\Omega) \) for \( j = 1, 2, \ldots \), and assume that \( f_j \to f \) uniformly on compact subsets of \( \Omega \) as \( j \to \infty \). Then either \( f \in \mathcal{O}^*(\Omega) \) of \( f \) is constantly equal to zero.

\textbf{Proof.} Same proof as for Theorem 3.11.

\textbf{Proposition 3.14.} Let \( \Omega \subset \mathbb{C} \) be a bounded \( \mathcal{C}^1 \)-smooth domain, let \( f \in \mathcal{O}(\Omega) \) \( \cap \mathcal{C}^1(\Omega) \), and assume that \( f(z) \neq 0 \) for all \( z \in b\Omega \). Then

\begin{equation}
2\pi i \sum_{a \in \Omega} \text{ord}_a(f) = \int_{b\Omega} \frac{f'(z)}{f(z)} \, dz.
\end{equation}

\textbf{Proof.} Set \( Z(f) = \{ a_1, \ldots, a_m \} \), choose \( \epsilon > 0 \) such that the closure of the disks \( D_\epsilon(a_j) \) are pairwise disjoint and contained in \( \Omega \). Then

\begin{equation}
\int_{b\Omega} \frac{f'(z)}{f(z)} \, dz = \sum_{j=1}^{m} \int_{bD_\epsilon(a_j)} \frac{f'(z)}{f(z)} \, dz.
\end{equation}

For each \( j \) we may write \( f(z) = (z-a_j)^{k(j)} \cdot g(z) \) with \( g(a_j) \neq 0 \), where \( k(j) = \text{ord}_{a_j}(f) \). So \( \frac{f'(z)}{f(z)} = \frac{k(j)}{(z-a_j)} + \frac{g'(z)}{g(z)} \), and so by possibly having to decrease \( \epsilon \) we see that \( \int_{bD_\epsilon(a_j)} \frac{f'(z)}{f(z)} \, dz = \int_{bD_\epsilon(a_j)} \frac{k(j)}{z-a_j} \, dz = 2\pi ik(j) \).
THEOREM 3.15. (Rouchet) Let \( \Omega \subset \mathbb{C} \) be a bounded \( C^1 \)-smooth domain, and let \( f \in \mathcal{O}(\Omega) \cap C^1(\overline{\Omega}) \), \( f(z) \neq 0 \), \( z \in b\Omega \). If \( g \in \mathcal{O}(\Omega) \cap C^1(\overline{\Omega}) \) and if \( |g(z)| < |f(z)| \) for all \( z \in b\Omega \), then

\[
\sum_{z \in \Omega} \text{ord}_z(f) = \sum_{z \in \Omega} \text{ord}_z(f + g).
\]

PROOF. Note that the functions \( h_t := f + t \cdot g, t \in [0, 1] \) are all nonzero on \( b\Omega \) by the assumption. By (3.7) the function \( \varphi(t) = \sum_{z \in \Omega} \text{ord}_z(h_t) \) is continuous on \([0, 1]\) and integer valued, hence the result follows. \( \square \)

PROPOSITION 3.16. Let \( \Omega \subset \mathbb{C} \) be open, let \( f \in \mathcal{O}(\Omega) \), and assume that \( f \) is injective. Then \( f'(z) \neq 0 \) for all \( z \in \Omega \).

PROOF. Fix \( a \in \Omega \). Without loss of generality we assume \( a = f(a) = 0 \) and write \( f(z) = \sum_{j \geq k} c_j(z - a)^j, c_k \neq 0 \). Choose \( r > 0 \) such that \( Z(f) \cap \overline{D}_r = \{0\} \) and such that \( Z(f') \cap \overline{D}_r \) is non-empty or the origin. For \( |c| < \|f\|_{bD_r} \) we have that \( \sum_{z \in D_r} \text{ord}_z(f - c) = k \). So there are points \( a_1, ..., a_k \in D_r \) with \( f(a_j) = c \), where the \( a_j \)'s a priori are not necessarily distinct. But \( f'(a_j) \neq 0 \) for each \( j \), so by the inverse function theorem \( f \) is injective near \( a_j \) for each \( j \). So the \( a_j \)'s are all distinct, hence \( k = 1 \). \( \square \)

PROPOSITION 3.17. Let \( \Omega \) be a domain, let \( f_j \in \mathcal{O}(\Omega) \), and assume that \( f_j \to f \) uniformly on compacts in \( \Omega \). If each \( f_j \) is injective, then either \( f \) is constant or \( f \) is injective.

PROOF. Assume to get a contradiction that there are two distinct points \( a_1, a_2 \in \Omega \) with \( f(a_1) = f(a_2) = 0 \). Choose a smoothly bounded domain \( \Omega' \subset \subset \Omega \) with \( f(z) \neq 0 \) for all \( z \in \Omega' \). If \( j \) is large enough we have that \( \sum_{z \in \Omega'} \text{ord}_z(f_j) = \sum_{z \in \Omega} \text{ord}_z(f) > 1 \).

PROPOSITION 3.18. Let \( \Omega \subset \mathbb{C} \) be open, let \( f \in \mathcal{O}(\Omega) \), and assume that \( f \) is injective. Then \( f'(z) \neq 0 \) for all \( z \in \Omega \).

PROOF. Fix \( a \in \Omega \). Without loss of generality we assume \( f(a) = 0 \) and write \( f(z) = \sum_{j \geq k} c_j(z - a)^j, c_k \neq 0 \). If \( r > 0 \) is small enough we have that \( |\sum_{j \geq k+1} c_j(z - a)^j| < c_k(z - a)^k \), and so \( \text{ord}_a(f) = \text{ord}_a((z - a)^k) \). Since \( f \) is injective we have \( k = 1 \), and \( f'(z) = c_1 \neq 0 \). \( \square \)

PROPOSITION 3.19. Let \( \Omega \subset \mathbb{C} \), let \( f \in \mathcal{O}(\Omega) \), and assume that \( f \) is injective. Then \( f^{-1} \in \mathcal{O}(\Omega) \).

PROOF. For any point \( a \in \Omega \) it follows from the inverse mapping theorem that \( df^{-1}(a) = \frac{1}{f'(a)}dz \). \( \square \)

THEOREM 3.20. (Laurent Series Expansion) Let \( f \in \mathcal{O}(A(r, s)) \) where \( A(r, s) := \{\zeta \in \mathbb{C} : r < |\zeta| < s\}, 0 \leq r < s \leq \infty \). Then there is a unique sequence \( c_j \in \mathbb{Z} \) such that

\[
f(\zeta) = \sum_{j=-\infty}^{\infty} c_j \zeta^j, \zeta \in A(r, s),
\]
and for any \( r < \rho < s \) we have that

\[
(3.11) \quad c_j = \frac{1}{2\pi i} \int_{bD_\rho} f(z) z^{-j-1} \, dz
\]

**Proof.** Choose \( r < \rho_1 < \rho_2 < s \). We have that

\[
(3.12) \quad f(\zeta) = \frac{1}{2\pi i} \int_{bD_{\rho_2}} \frac{f(z)}{z - \zeta} \, dz - \frac{1}{2\pi i} \int_{bD_{\rho_1}} \frac{f(z)}{z - \zeta} \, dz.
\]

We have considered the first integral in the proof that holomorphic functions are analytic, so let’s look at the second. We have that \(-1/(z - \zeta) = 1/\zeta (1 - (z/\zeta)) = 1/\zeta \sum_{j=0}^{\infty} (z/\zeta)^j = \sum_{j=0}^{\infty} z^j \zeta^{-j-1} \) for \(|\zeta| > |z|\). By uniform convergence as in the proof that holomorphic functions are analytic we may interchange summation and integration and get that

\[
(3.13) \quad -\frac{1}{2\pi i} \int_{bD_{\rho_1}} \frac{f(z)}{z - \zeta} \, dz = \sum_{j=0}^{\infty} \left( \frac{1}{2\pi} \int_{bD_{\rho_1}} f(z) z^j \right) \zeta^{-j-1}
\]

for \(|\zeta| > \rho_1\). By Stoke’s Theorem the formula for the \( c_j \)'s is independent of the choice of \( \rho_1, \rho_2 \). Uniqueness follows from the residue theorem. \( \square \)

**Theorem 3.21.** Let \( f \in \mathcal{O}(\mathbb{D}^*) \) and assume that \( f \) is bounded. Then \( f \) extends to a holomorphic function on \( \mathbb{D} \).

**Proof.** We consider the Laurent series coefficients \( c_j \) for \( j < 0 \) and get that

\[
(3.14) \quad 2\pi i c_j = \int_{bD_{\zeta}} f(z) z^{-j-1} \, dz = \int_{0}^{2\pi} f(e^{i\theta}) i e^{i\theta} \, d\theta \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0
\]

since \( |f| \) is bounded. By Abel’s Lemma we have that \( f \) extends to \( \mathbb{D} \). \( \square \)
CHAPTER 2

Runge’s Theorem

1. Partitions of unity

This can be read in Narasimhan’s book, Chapter 5., Section 1.

2. Smoothened Cauchy Integral Formula

THEOREM 2.1. Let \( \Omega \subset \mathbb{C} \) be a domain, let \( f \in \mathcal{O}(K) \), i.e., there exists an open neighborhood \( U \subset \Omega \) of \( K \) with \( f \in \mathcal{O}(U) \), and let \( \alpha \in \mathcal{C}_0^\infty(U) \) with \( \alpha \equiv 1 \) near \( K \). Then for all \( \zeta \in K \) we have that

\[
 f(\zeta) = -\frac{1}{\pi} \int \int_{\mathbb{C}} \frac{\partial \alpha(z) \cdot f(z)}{z - \zeta} \, dx \, dy, \quad z = x + iy.
\]

Proof. This is immediate from the generalized Cauchy Integral Formula, since \( \alpha \cdot f \) is compactly supported in \( \mathbb{C} \) and \( \frac{\partial}{\partial z}(\alpha \cdot f) = \frac{\partial \alpha}{\partial z} \cdot f \) since \( f \) is holomorphic.

\[ \square \]

3. Runge’s Theorem

DEFINITION 3.1. Let \( \Omega \subset \mathbb{C} \) be a domain, and let \( U \subset \Omega \) be a subset. We say that \( U \) is relatively compact in \( \Omega \) and write \( U \subset\subset \Omega \), if \( \operatorname{cl}_\mathbb{C}(U) \) is compact.

LEMMA 3.2. Let \( \Omega \subset \mathbb{C} \) be a domain, and let \( K \subset \Omega \) be compact. Let \( U \) be a connected component of \( \Omega \setminus K \) and let \( \tilde{U} \) be the connected component of \( \mathbb{C} \setminus K \) containing \( U \). Then the following are equivalent

1. \( U \subset\subset \Omega \),
2. \( U \) is bounded and \( U = \tilde{U} \), and
3. \( U \) is bounded and \( \partial_C U \subset K \).

Proof. (1) \( \Rightarrow \) (2) Clearly \( U \) is bounded, and if \( U \neq \tilde{U} \) there is a sequence of points \( \{z_j\} \subset U \) converging to a point \( z \in \tilde{U} \setminus U \), hence \( \partial_C U \) is not compact.

(2) \( \Rightarrow \) (3) Let \( z \in bU \). If \( z \notin K \) there exists a disk \( D_r(z) \cap K = \emptyset \), hence \( \tilde{U} \neq U \).

(3) \( \Rightarrow \) (1) \( b_C U \subset K \Rightarrow cl_C(U) \subset \Omega \Rightarrow cl_C(U) = cl_\mathbb{C}(U) \), and we know that a closed and bounded set is compact.

\[ \square \]
DEFINITION 3.3. Let $\Omega \subset \mathbb{C}$ be a domain and let $K \subset \Omega$ be compact. We set
\begin{equation}
\hat{K}_{O}(\Omega) := \{z \in \Omega : |f(z)| \leq \|f\|_K \text{ for all } f \in O(\Omega)\}.
\end{equation}
The set $\hat{K}_{O}(\Omega)$ is called the holomorphically convex hull of $K$.

Lemm 3.4. Let $\Omega \subset \mathbb{C}$ be a domain and let $K \subset \Omega$ be compact. Then $\hat{K}_{O}(\Omega)$ is compact.

Proof. It is clear that $\hat{K}_{O}(\Omega)$ is a closed subset of $\Omega$. Since $K \subset B_R$ for a large enough $R > 0$ is also clear that $\hat{K}_{O}(\Omega)$ is bounded. To see that $\hat{K}_{O}(\Omega)$ is bounded, for any point $a \in b \Omega$ we have that $f_a(z) = \frac{1}{z-a}$ satisfies $\|f_a\|_K \leq \frac{1}{\delta}$, and for any point in the set $S := \{z \in \Omega : \text{dist}(z, b\Omega) \leq \frac{1}{2}\}$ there exists an $f_a$ with $|f_a(z)| \geq \frac{2}{\delta}$. □

Proposition 3.5. (Pushing poles) Let $K \subset \mathbb{C}$ be a compact set and let $U \subset \mathbb{C} \setminus K$ be a connected component. Let $a \in U$ be any point, and let $\tilde{U}$ denote the set of points $b \in U$ such that the function $f_a(z) = \frac{1}{z-a}$ may be approximated uniformly on $K$ by functions of the form
\begin{equation}
f_b(z) = \sum_{j=-N}^{N} c_j(z-b)^j.
\end{equation}
Then $\tilde{U} = U$.

Proof. By assumption we have that $\tilde{U}$ is non-empty. It is not hard using Laurent series expansion to show that $\Omega$ is also open and closed (Do it!). □

Corollary 3.6. Let $\Omega \subset \mathbb{C}$ be a domain, and let $K \subset \Omega$ be a compact set. Then $\hat{K}_{O}(\Omega)$ is the union of $K$ and all the components of $\Omega \setminus K$ which are relatively compact in $\Omega$.

Proof. If $U \subset \Omega \setminus K$ is relatively compact in $\Omega$ it follows form Lemma 3.2 and the maximum principle that $U \subset \hat{K}_{O}(\Omega)$. On the other hand, let $U \subset \Omega \setminus K$ not be relatively compact in $\Omega$ and let $a \in U$. Then $U \setminus \{a\}$ is a connected component of $\Omega \setminus \{K \cup \{a\}\}$ which is not relatively compact. Hence, by Lemma 3.2 and Propositon 3.5 any $z \mapsto \frac{1}{z-a}, b \in U \setminus \{a\}$ may be approximated on $K$ by functions holomorphic on $\Omega$. Considering $b$ close enough to $a$ this shows that $a \notin \hat{K}_{O}(\Omega)$. □

Proposition 3.7. Let $K \subset \mathbb{C}$ be a compact set, and let $f \in O(K)$. For any $\epsilon > 0$ there exist $a_j, c_j \in \mathbb{C} \setminus K, j = 1, ..., N = N(\epsilon)$ such that the function
\begin{equation}
r(z) = \sum_{j=1}^{N} \frac{c_j}{z-a_j}.
\end{equation}
satisfies $\|r - f\|_K < \epsilon$.

**Proof.** Let $U$ be an open set containing $K$ such that $f \in \mathcal{O}(U)$ and let $\alpha \in C^\infty_c(U)$ with $\alpha \equiv 1$ near $K$. Write $g(z) := \frac{\partial \alpha}{\partial z} \cdot f(z)$ and choose an open set $S$ with $\overline{S}$ disjoint from $K$ such that $\text{Supp}(g) \subset S$. It follows from Theorem 2.1 that

$$f(\zeta) = -\frac{1}{\pi} \int \int_S \frac{g(z)}{z-\zeta} dx dy,$$

for all $\zeta \in K$. Choose a sequence $\{\Delta_k^j\}, j \in \mathbb{N}, 1 \leq k \leq m(j)$ of disjoint open squares of radius $r(k,j) \leq 1/j$ whose closures cover $\text{Supp}(g)$ and pick a point $z_{jk}$ in each square. For each fixed $\zeta$ we know that the sequence of Riemann sums

$$R_j(\zeta) := \sum_{k=1}^{m(j)} \frac{g(z_{jk})}{z_{jk} - \zeta}$$

converges uniformly to $f(\zeta)$ for $\zeta \in K$ as $j \to \infty$. By compactness of $K$ and since $\overline{S}$ is disjoint from $K$ it follows that the convergence is uniform independently of $\zeta \in K$. \hfill \Box

**Theorem 3.8.** (Runge’s Theorem) Let $\Omega \subset \mathbb{C}$ be a domain and let $K \subset \Omega$ be compact. The following are equivalent:

1. $\mathcal{O}(\Omega)$ is dense in $\mathcal{O}(K)$,
2. $\Omega \setminus K$ has no relatively compact components in $\Omega$, and
3. $\hat{K}_{\mathcal{O}(\Omega)} = K$.

**Proof.** The equivalence $(2) \Leftrightarrow (3)$ is Corollary 3.6. We show first $(2) \Rightarrow (1)$. By Proposition 3.7 it is enough to show that we may approximate the function $r(z) = \frac{1}{z-a}$ for any point $a \in \Omega \setminus K$. There are two possibilities: (i) $a$ is in a bounded connected component $U$ of $\mathbb{C} \setminus K$. By Lemma 3.2 there is a point in $U \setminus \Omega$ so we can use Proposition 3.5. (ii) $a$ is in the unbounded component of $\mathbb{C} \setminus K$. By Proposition 3.5 we may assume that $|a| > |z|_K$, and then it follows by Taylor series expansion.

Finally we show $(1) \Rightarrow (2)$. Suppose that $(2)$ does not hold. Then by Lemma 3.2 there is a connected component $U$ of $\Omega \setminus K$ with $b_C U \subset K$. Pick a point $a \in U$ and define $r(z) = \frac{1}{z-a}$. Suppose there exists a sequence $f_j \in \mathcal{O}(\Omega)$ such that $f_j \to r(z)$ uniformly on $K$. Then by the maximum principle $f_j \to f \in \mathcal{O}(U) \cap C(\overline{U})$ with $f = r$ on $b U$. So $g = (z-a) \cdot f$ is a holomorphic function on $U$ which is identically one on $U$, hence $g \equiv 1$. This is a contradiction since $g(a) = 0$. \hfill \Box

3.0.1. **Runge’s Theorem for non-vanishing holomorphic functions.**

**Theorem 3.9.** Let $\Omega \subset \mathbb{C}$ be a domain, and let $K \subset \Omega$ be a compact set, $\hat{K}_{\mathcal{O}(\Omega)} = K$. Then for any $f \in \mathcal{O}^\ast(K)$ and any $\epsilon > 0$ there exists $F \in \mathcal{O}^\ast(\Omega)$
with
\begin{equation}
\|F - f\|_K < \epsilon.
\end{equation}

**Proof.** Using Proposition 3.7 we will now assume that \( f \) is of the form
\begin{equation}
f(z) = \sum_{j=1}^{N} \frac{c_j}{z - a_j} = c \cdot \Pi_{j=1}^{M} (z - b_j)^{m_j},
\end{equation}
with \( c, b_j \in \mathbb{C}, b_j \notin K, m_j \in \mathbb{Z} \). It is then enough to show that each function \( (z - b_j) \) can be approximated arbitrarily well on \( K \) by non-zero holomorphic functions on \( \Omega \).

The idea is as follows: Ideally, if \( \log(z - b_j) \) exists near \( K \) we could approximate \( \log(z - b_j) \) by some function \( g \) using the ordinary Runge's Theorem, and then use \( e^g \) as an approximation of \( \log(z - b_j) \) on \( K \). Such a logarithm does not exist in general, so we will do the following: since \( K \) is holomorphically convex we are in one of two situations, either (i) there exists a point \( d_j \) in the same connected component of \( \mathbb{C} \setminus K \) as \( b_j \) with \( d_j \notin \Omega \) (Lemma 3.2), or (ii) there is a point \( d_j \) in the same connected component of \( \mathbb{C} \setminus K \) as \( b_j \) with \( |d_j| > \|z\|_K \). In any case, we may write
\begin{equation}
(z - b_j) = \left( \frac{z - b_j}{z - d_j} \right) \cdot (z - d_j).
\end{equation}
So it is enough to show that we may approximate both factors in the right hand side product but non-vanishing holomorphic functions. In situation (i) the function \( (z - d_j) \) is already non-vanishing on \( \Omega \), and in situation (ii) the function \( \log(z - d_j) \) exists near \( K \), so it is enough to show that the first factor may be approximated. For this it is enough to show that \( \log(z - b_j) \) exists near \( K \). To see this we show the following:

**Lemma 3.10.** Let \( f_t : K \rightarrow \mathbb{C}^* \) be a homotopy of continuous maps, \( t \in [0, 1] \). If \( \arg(f_0) \) exists on \( K \) then \( \arg(f_t) \) exists on \( K \) for all \( t \in [0, 1] \).

**Proof.** For each \( t \) and each \( z \in K \) we define \( \arg(f_t(z)) \) to be the angle you get by continuing \( \arg \) from \( \arg(f_0(z)) \). We need to show that this function is continuous for each \( t \). By compactness there exists a \( \delta > 0 \) such that the following holds: for any \( t \in [0, 1] \) and any \( z \in K \), the difference \( |\arg(f_t(z)) - \arg(f_{t'}(z))| < \pi/2 \) for all \( |t - t'| \leq \delta \), if \( \arg(f_{t'}(z)) \) is the branch you get by continuing from \( \arg(f_t(z)) \). It follows that \( \arg(f_{t'}) \) is continuous if \( \arg(f_t) \) is: we get
\[
|\arg(f_{t'}(z)) - \arg(f_{t'}(z'))| \leq |\arg(f_t'(z) - \arg(f_t)(z))|
+ |\arg(f_t(z)) - \arg(f_t(z'))|
+ |\arg(f_t(z')) - \arg(f_t'(z'))|
\leq |\arg(f_t(z)) - \arg(f_t(z'))| + \pi,
\]
from which the result follows since \( \arg \) is always continuous modulo \( 2\pi \). \( \square \)
The claim that \( \log\left( \frac{z-b_j}{z-d_j} \right) \) exists now follows from the fact that \( b_j \) and \( d_j \) lie in the same path-connected component of \( \mathbb{C} \setminus K \). \qed
CHAPTER 3

Applications of Runge’s Theorem

1. The $\bar{\partial}$-equation

**Proposition 1.1.** Let $\Omega \subset \mathbb{C}$ be a domain, and let $u \in \mathcal{C}^1_0(\Omega)$. Then there exists $f \in \mathcal{C}^1(\Omega)$ solving the equation

\begin{equation}
\frac{\partial f}{\partial \bar{z}}(\zeta) = u(\zeta), z = x + iy.
\end{equation}

for all $\zeta \in \Omega$.

**Proof.** We define

\begin{equation}
f(\zeta) := -\frac{1}{\pi} \int_{\mathbb{C}} \frac{u(z)}{z - \zeta} dx dy, z = x + iy.
\end{equation}

for $\zeta \in \mathbb{C}$. This is well defined since $u$ has compact support. Note that

\begin{equation}
f(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{u(z + \zeta)}{z} dx dy
\end{equation}

Differentiating with respect to the $x$-variable we consider real $\delta$’s, and we get that

$$\lim_{\delta \to 0} \frac{f(\zeta + \delta) - f(\zeta)}{\delta} = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\delta} \frac{1}{z} u(z + \zeta + \delta) - u(z + \zeta) dx dy$$

$$= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{\delta} \frac{1}{z} u(z + \zeta + \delta) - u(z + \zeta) dx dy$$

$$= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial u}{\partial \bar{z}}(z) dx dy.$$

Differentiating with respect to the $y$-variable can be computed similarly, and so we get that

\begin{equation}
\frac{\partial f}{\partial \bar{z}}(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial u}{\partial \bar{z}}(z) dx dy = u(\zeta),
\end{equation}

where the last equality follows from the generalized Cauchy Integral Formula. $\square$

**Theorem 1.2.** Let $\Omega \subset \mathbb{C}$ be a domain, and let $u \in \mathcal{C}^1(\Omega)$. Then there exists $f \in \mathcal{C}^1(\Omega)$ satisfying the equation

\begin{equation}
\frac{\partial f}{\partial \bar{z}}(\zeta) = u(\zeta),
\end{equation}

for all $\zeta \in \Omega$. 19
PROOF. Let $K_j \subset K_{j+1}$ be a normal exhaustion of $\Omega$. For each $j$ let $\alpha_j \in C_0^\infty(\Omega)$ such that $\alpha_j \equiv 1$ near $K_j$. Let $u_j := \alpha_j \cdot u$. We will solve $\frac{\partial f_j}{\partial \bar{z}} = u_j$ by induction. Assume that we have solved $\frac{\partial f_k}{\partial \bar{z}} = u_k$. Let $\tilde{f}_{k+1}$ be a solution to the equation $\frac{\partial \tilde{f}_{k+1}}{\partial \bar{z}} = u_{k+1}$. Then $\tilde{f}_{k+1} - f_k \in O(K_m)$ so by Runge’s Theorem there exists $g_{k+1} \in O(\Omega)$ with $\| \tilde{f}_{k+1} - f_k - g_{k+1} \|_{K_m} < (1/2)^k$. We set $f_{k+1} := \tilde{f}_{k+1} - g_{k+1}$. Now it is clear that $\{f_j\}$ is a Cauchy sequence on each $K_i$ and so $f_j \to f \in C(\Omega)$. For each fixed $i$ we see that $f - f_i \in O(K_i^\circ)$, and so $f \in C^1(\Omega)$ and solves our equation. □

2. The theorems of Mittag-Leffler and Weierstrass

Theorem 2.1. (Mittag-Leffler) Let $\Omega \subset \mathbb{C}$ be a domain, let $A = \{a_j\}$ be a discrete set of points, and for each $j \in \mathbb{N}$ let $p_j$ be a prescribed principle part at $a_j$

\begin{align}
(2.1) \quad p_j(z) = \sum_{j=1}^{m(j)} c_j \cdot (z - a_j)^{-j}.
\end{align}

Then there exists $f \in O(\Omega \setminus A)$ such that $f - p_j$ is holomorphic near $a_j$ for all $j$.

We will leave it as an exercise to prove this theorem in a similar manner as the previous theorem, and we give here two different proofs.

Proof no. 1 of Theorem 2.1: Choose pairwise disjoint disks $D_{\delta(a)}(a) \subset \Omega, a \in A$, and let $\phi_a \in C_0^\infty(D_{\delta(a)}(a))$ be constantly equal to 1 near $a$. Then $f := \sum_a \phi_a p_a$ is a smooth solution to our problem. Now $\frac{\partial f}{\partial \bar{z}}$ extends to a smooth function $u$ on $\Omega$ and we may solve $\bar{\partial} g = u$ on $\Omega$. So $f - g$ is a holomorphic solution. □

Lemma 2.2. Let $\Omega \subset \mathbb{R}^n$ be a domain, and let $U = \{U_\alpha\}_{\alpha \in I}$ be an open covering of $\Omega$. Furthermore, let $f_{\alpha \beta} \in C^\infty(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in I$ such that

\begin{align}
(2.2) \quad f_{\alpha \beta} + f_{\beta \gamma} + f_{\gamma \alpha} = 0 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma \text{ for all } \alpha, \beta, \gamma \in I.
\end{align}

Then there exist $f_\alpha \in C^\infty(U_\alpha)$ for all $\alpha \in I$ such that $f_{\alpha \beta} = f_\alpha - f_\beta$ on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$.

Proof. Let $\{\phi_\alpha\}$ be a partition of unity with respect to the cover $U$. We define $f_\alpha := \sum_{\gamma \in I} \phi_\gamma \cdot f_{\alpha \gamma}$ and note that $f_\alpha \in C^\infty(U_\alpha)$ since $f_{\alpha \alpha} = 0$ by
2. THE THEOREMS OF MITTAG-LEFFLER AND WEIERSTRASS

On $U_\alpha \cap U_\beta$ we have that

$$f_\alpha - f_\beta = \sum_\gamma \phi_\gamma f_{\alpha \gamma} - \phi_\gamma f_{\beta \gamma}$$

$$= \sum_\gamma \phi_\gamma (f_{\alpha \gamma} + f_{\gamma \beta})$$

$$= \sum_\gamma \phi_\gamma f_{\alpha \beta}$$

$$= f_{\alpha \beta}.$$

\[\Box\]

**Theorem 2.3.** Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of $\Omega$. Furthermore, let $f_{\alpha \beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in I$ such that

$$f_{\alpha \beta} + f_{\beta \gamma} + f_{\gamma \alpha} = 0$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ for all $\alpha, \beta, \gamma \in I$.

Then there exist $f_\alpha \in \mathcal{O}(U_\alpha)$ for all $\alpha \in I$ such that $f_{\alpha \beta} = f_\alpha - f_\beta$ on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$.

**Proof.** We have seen that there exist $f_\alpha \in C^\infty(U_\alpha)$ which solve the problem. Then $f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in I$ and so $u := \overline{\partial}f_\alpha$ is well defined on $\Omega$. Let $f$ solve $\overline{\partial}f = u$. Then $f_\alpha - f$ solves the problem. \[\Box\]

**Proof no. 2 of Theorem 2.1:** For each $a \in A$ let $U_a = \Omega \setminus (A \setminus \{a\})$. On $U_a \cap U_b$ set $f_{ab} = p_a - p_b$. Then $f_{ab} \in \mathcal{O}(U_a \cap U_b)$, and clearly $f_{ab} + f_{bc} + f_{ca} = 0$. Let $f_\alpha \in \mathcal{O}(U_\alpha)$ for all $a$ such that $f_{ab} = f_a - f_b$. Define $f := p_a - f_a$ on $U_a$ and note that $f$ is now well defined. \[\Box\]

**Theorem 2.4.** (Weierstrass) Let $\Omega \subset \mathbb{C}$ be a domain, let $A = \{a_j\}$ be discrete, and for each $j \in \mathbb{N}$ let $m_j \in \mathbb{Z}$. Then there exists $f \in \mathcal{O}^*(\Omega \setminus A)$ such that $f \cdot (z - a_j)^{-m_j}$ is holomorphic and nonzero near $a_j$ for all $j$.

**Proof.** Copy the proof of Theorem 2.1 based on the proof of Theorem 1.2 using products instead of sums, and Runge’s theorem for non-zero approximation.

Alternatively, copy the second proof of Theorem 2.1 using Theorem 2.5 instead of Theorem 2.3. \[\Box\]

**Theorem 2.5.** Let $\Omega \subset \mathbb{C}$ be a domain, and let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open covering of $\Omega$ by simply connected domains. Furthermore, let $f_{\alpha \beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ for all $\alpha, \beta \in I$ such that

$$f_{\alpha \beta} \cdot f_{\beta \gamma} \cdot f_{\gamma \alpha} = 1$$

on $U_\alpha \cap U_\beta \cap U_\gamma$ for all $\alpha, \beta, \gamma \in I$.

Then there exist $f_\alpha \in \mathcal{O}^*(U_\alpha)$ for all $\alpha \in I$ such that $f_{\alpha \beta} = f_\alpha / f_\beta$ on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in I$. \[\Box\]
Proof. From topology we know that there exists a continuous solution \( \{g_\alpha\} \). For each \( \alpha \) choose a branch \( h_\alpha = \log g_\alpha \), and set \( \tilde{f}_{\alpha\beta} = h_\alpha - h_\beta = \log f_{\alpha\beta} \). Then \( \tilde{f}_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta) \) and the condition (2.4) is satisfied. By Theorem 2.3 there are functions \( g_\alpha \in \mathcal{O}(U_\alpha) \) such that \( \tilde{f}_{\alpha\beta} = g_\alpha - g_\beta \). So \( f_\alpha := e^{g_\alpha} \) solves the problem. \( \square \)
CHAPTER 4

Piccard’s Theorem

This can be read in Narasimhan’s book, Chapter 4.
CHAPTER 5

Riemann mapping theorem

This can be read in Narasimhan’s book, Chapter 7, pp 139–144.
CHAPTER 6

Some exercises fro Narasimhan/Nievergelt

1. DeRham-Hodge Theorem

The goal of this section is to show that the genus $g_X = \dim H^1(X, \mathcal{O})$ of a compact Riemann surface $X$ is a topological invariant, i.e., if $X$ is homeomorphic to $X'$ then $g_X = g_{X'}$. This is done by finding a basis consisting of holomorphic and anti-holomorphic 1-forms for the first DeRham cohomology group. Consider the short exact sequence

$0 \to \mathbb{C} \to E \xrightarrow{d} \mathcal{E}^{(1)}_{cl} \to 0,$

on $X$. Since $H^1(X, \mathcal{E}) = 0$ we have seen that

$H^1(X, \mathbb{C}) \approx \mathcal{E}^{(1)}_{cl}(X)/d\mathcal{E}(X),$

where the quotient on the right hand side is the first DeRham group, denoted by $Rh^1(X)$. By (1.2) the dimension of $Rh^1(X)$ is a topological invariant.

Recall that for a Riemann surface $X$, $\Omega$ denotes the sheaf of holomorphic 1-forms, and $\overline{\Omega}$ denotes the sheaf of anti-holomorphic 1-forms. We define the sheaf of harmonic 1-forms to be the direct sum

$Harm^1 := \Omega \oplus \overline{\Omega}.$

Note that by Serre duality we have that

$\dim Harm^1(X) = 2g.$
Theorem 1.1. Let $X$ be a compact Riemann surface. Then

\begin{equation}
H^1(X, \mathbb{C}) \approx \text{Rh}^1(X) \approx \text{Harm}^1(X).
\end{equation}

Note that by (1.4) and (1.2) the genus of $X$ is a topological invariant.

Proof. We have seen that $H^1(X, \mathcal{O}) \approx \mathcal{E}^{(0,1)}(X)/\overline{\partial}\mathcal{E}(X)$, so in particular, the dimension of the latter group is $g$. By Serre duality we have that $\dim \Omega(X) = g$. Moreover, no non-trivial form $\omega \in \Omega$ is $\overline{\partial}$-exact: if $\overline{\partial}f = \omega \in \Omega$ then $\partial \overline{\partial} f = 0$ and so $f$ is harmonic. It follows that

\begin{equation}
\mathcal{E}^{(0,1)}(X) = \overline{\partial}\mathcal{E}(X) \oplus \Omega(X).
\end{equation}

By taking complex conjugates and considering the direct sum we get that

\begin{equation}
\mathcal{E}^{(1)}(X) = \partial\mathcal{E}(X) \oplus \overline{\partial}\mathcal{E}(X) \oplus \Omega(X) \oplus \overline{\Omega}(X).
\end{equation}

To describe $\mathcal{E}^{(1)}_{cl}(X)$ let $\omega = \partial f + \overline{\partial} g + \omega_1 + \overline{\omega}_2$ be a $d$-closed form decomposed according to (1.7). Since $\omega_1$ and $\overline{\omega}_2$ are both $d$-closed it follows that $\partial f + \overline{\partial} g$ is $d$-closed, i.e., $\partial \overline{\partial} f + \overline{\partial} \partial g = 0$. So $f - g$ is harmonic, hence constant. It follows that $df = \partial f + \overline{\partial} f = \partial f + \overline{\partial}(g + c) = \partial f + \overline{\partial} g$, hence $\omega = df + \omega_1 + \overline{\omega}_2$. It follows that

\begin{equation}
\mathcal{E}^{(1)}_{cl}(X) = d\mathcal{E}(X) \oplus \Omega(X) \oplus \overline{\Omega}(X).
\end{equation}

Since no nontrivial element of $\Omega(X) \oplus \overline{\Omega}(X)$ is $d$-exact (why?) the proof is complete. \qed