Elementary Theory of Holomorphic Functions

1 Some basic properties of $\mathbb{C}$-differentiable and holomorphic functions

1.1 Complex derivatives and Cauchy–Riemann equations

The following exercises extend differential calculus to complex analysis.

Exercise 37. Prove that for all $c_0, c_1 \in \mathbb{C}$ the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) := c_0 + c_1 z$ has a complex derivative, with $f'(z) = c_1$ for every $z \in \mathbb{C}$.

Exercise 38. Prove that for all open sets $\mathcal{D}_f \subseteq \mathbb{C}$ and $\mathcal{D}_g \subseteq \mathbb{C}$, and all functions $f : \mathcal{D}_f \to \mathbb{C}$ and $g : \mathcal{D}_g \to \mathbb{C}$ that both have a complex derivative at $z \in \mathcal{D}_f \cap \mathcal{D}_g$,

$$\begin{aligned} (f+g)'(z) &= f'(z) + g'(z), \\ (fg)'(z) &= f'(z)g(z) + f(z)g'(z); \end{aligned}$$

moreover, if $g(z) \neq 0$, then

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}.$$

Exercise 39. Prove that for each positive integer n and for all $c_0, \ldots, c_n \in \mathbb{C}$, the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) := c_0 + c_1 z + c_2 z^2 + \cdots c_n z^n$ has a complex derivative, with $f'(z) = c_1 + 2c_2 z + \cdots n c_n z^{n-1}$ for every $z \in \mathbb{C}$.

Exercise 40. Prove that for all open sets $\mathcal{D}_f \subseteq \mathbb{C}$ and $\mathcal{D}_g \subseteq \mathbb{C}$, and all functions $f : \mathcal{D}_f \to \mathbb{C}$ with a complex derivative at $z \in \mathcal{D}_f$, and $g : \mathcal{D}_g \to \mathbb{C}$ with a complex derivative at $w := f(z) \in \mathcal{D}_g$,

$$(g \circ f)'(z) = \{g'[f(z)]\} \cdot f'(z).$$

Exercise 41. Prove that for all open sets $\mathcal{D}_\gamma \subseteq \mathbb{R}$ and $\mathcal{D}_f \subseteq \mathbb{C}$, and all functions $\gamma : \mathcal{D}_\gamma \to \mathbb{C}$ with a real derivative at $t \in \mathcal{D}_\gamma$, and $f : \mathcal{D}_f \to \mathbb{C}$ with a complex derivative at $z := \gamma(t) \in \mathcal{D}_f$, the composite function $f \circ \gamma$ has a real derivative at t given by

$$(f \circ \gamma)'(t) = \{f'[\gamma(t)]\} \cdot \gamma'(t).$$

Exercise 42. (This result is for Koebe's Prop. 1, §2, Ch. 7.) For all $p, q \in \mathbb{C}$, prove that there does *not* exist any $c \in \mathbb{C}$ such that for every z with $qz^2 \neq 1$,

$$\frac{z^2 + p}{1 - qz^2} = c \cdot z.$$

Exercise 43. Prove that if $f'(z)$ exists, then f is continuous at z.

Exercise 44. Prove that if $f : \mathcal{D} \to \mathbb{C}$ has all its first partial derivatives defined at any point in the open set $\mathcal{D} \subseteq \mathbb{C}$, then at that point

$$\frac{\partial}{\partial z} f = \overline{\frac{\partial}{\partial \overline{z}} \overline{f}}.$$

Exercise 45. Prove that if $f : \mathbb{C} \to \mathbb{C}$ satisfies the Cauchy–Riemann equations at z, then the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(w) := \overline{f(\overline{w})}$ satisfies the Cauchy–Riemann equations at $w := \overline{z}$.

Exercise 46. Prove that if $u : V \subseteq \mathbb{C} \to \mathbb{R}$ has continuous second partial derivatives, and if $f : U \subseteq \mathbb{C} \to \mathbb{C}$ has a complex derivative and continuous second partial derivatives everywhere in the open set U, then f' satisfies the Cauchy–Riemann equations, and

$$\Delta(u \circ f)_z = \Delta u_{f(z)} \cdot \left|f'(z)\right|^2.$$

Conclude that if u is harmonic in an open set $V \subseteq \mathbb{C}$, which means that $\Delta u = 0$ throughout V, then the composition $u \circ f : [U \cap f^{-1}(V)] \to \mathbb{R}$ is harmonic.

Definition 11. If f has a complex derivative at $z \in \mathbb{C}$, then the **differential of** f **at** z is the linear function $df_z : \mathbb{C} \to \mathbb{C}$ defined by $df_z(\zeta) := f'(z) \cdot \zeta$.

Also, the (complex) **line tangent to** f **at** z is the affine function $T_{f,\zeta} : \mathbb{C} \to \mathbb{C}$ defined by $T_{f,z}(w) := f(z) + df_z(w - z) = f(z) + f'(z) \cdot (w - z)$. □

Exercise 47. Prove that if f has a complex derivative at $z \in \mathbb{C}$, then the line tangent to f at z is the affine function closest to f near z, in the sense that for each affine function $L : \mathbb{C} \to \mathbb{C}$ defined by complex coefficients c_0 and c_1 with $L(\zeta) := c_0 + c_1\zeta$, there exists a real $\delta > 0$ such that for each number $w \in \mathbb{C}$ for which $0 < |w - z| < \delta$, the following inequality holds:

$$|f(w) - L(w)| \geq \left|f(w) - [f(z) + f'(z) \cdot (w - z)]\right|.$$

Exercise 48. For each function $f : \Omega \to \mathbb{C}$ holomorphic on a *connected* open set $\Omega \subseteq \mathbb{C}$ prove the following statements.

(48.1) If $f'(z) = 0$ for every $z \in \Omega$, then f is constant.

(48.2) If there exists $c \in \mathbb{C}$ such that $f(z) = c \cdot \overline{f(z)}$ for every $z \in \Omega$, then f is constant.

(48.3) If $f(\Omega) \subseteq \mathbb{R}$, then f is constant.

(48.4) If $|f|$ is constant, then f is constant.

(48.5) If $g : \mathbb{C} \to \mathbb{C}$ is holomorphic and $g \circ f$ is constant, then f or g is constant.

(48.6) If $f_1, \ldots, f_N$ are holomorphic on Ω, and if $|f_1|^2 + \cdots + |f_n|^2$ is constant, then each f_j is constant.

2.1 Complex line integrals

Exercise 74. Prove that for each function $f : \mathcal{D} \to \mathbb{C}$ continuous on an open s
$\mathcal{D} \subseteq \mathbb{C}$ and for each differentiable curve $\gamma : [0, 1] \to \mathcal{D}$,

$$\int_\gamma f(z)\, dz = \lim_{\substack{N \to \infty \\ \max|t_{k+1} - t_k| \to 0}} \sum_{k=0}^{N-1} f(z_k)(z_{k+1} - z_k)$$

for every sequence (z_k) on the image of γ such that $z_k = \gamma(t_k)$ and $0 = t_0 <$
$\cdots < t_{N-1} < t_N = 1$.

Exercise 75. Prove that if $f : \mathcal{D} \to \mathbb{C}$ has a complex derivative that is conti us
on the open set $\mathcal{D} \subseteq \mathbb{C}$, and if $\gamma : [0, 1] \to \mathcal{D}$ is a differentiable curv om
$z_0 := \gamma(0)$ to $z_1 := \gamma(1)$, then

$$f(z_1) - f(z_0) = \int_\gamma f'(\zeta)\, d\zeta.$$

Exercise 76. Prove that if $f : \mathcal{D} \to \mathbb{C}$ has a complex derivative f' that is in-
uous on the open set $\mathcal{D} \subseteq \mathbb{C}$, and if $\gamma : [0, 1] \to \mathbb{C}$ is a differentiable curv om
$z_0 := \gamma(0)$ to $z_1 := \gamma(1)$, then

$$|f(z_1) - f(z_0)| \leq \text{Length}(\gamma) \cdot \max\{|f'(z)| : z \in \gamma([0, 1])\}.$$

Exercise 77. Prove that if a differentiable curve $\gamma : [0, 1] \to \mathbb{C}$ paran zes
counterclockwise the boundary $\partial\Omega$ of an open set $\Omega \subset \mathbb{C}$, and if $A(\Omega)$ den the
area of Ω, then under suitable conditions on Ω,

$$A(\Omega) = \frac{1}{2i} \oint_{\partial\Omega} \bar{z}\, dz.$$

Exercise 78. For every closed rectangle R in an open subset $\Omega \subseteq \mathbb{R}^2$, and f very
continuously differentiable function $f : \Omega \to \mathbb{R}^2$, apply the following ve n of
Stokes' Theorem from the text,

$$\int\int_R \frac{\partial f}{\partial \bar{z}}\, dx\, dy = \frac{1}{2i} \oint_{\partial R} f(z)\, dz,$$

to design a proof that if $f \in C^1(\Omega)$ and $\oint_{\partial R} f(z)\, dz = 0$ for every closed r ngle
R in Ω, then f has a complex derivative everywhere in Ω.

Definition 22. For each complex number $c \in \mathbb{C}$, and for each differentiable closed curve $\gamma : [a, b] \to \mathbb{C}\backslash\{c\}$ that does not pass through the point c, define the **winding number** $n(\gamma, c)$ of the curve γ with respect to the point c by

$$n(\gamma, c) := \frac{1}{2\pi i} \oint_{\gamma} \frac{1}{z - c}\, dz. \qquad \square$$

Definition 23. Let $\sigma : [a, b] \to S^1 \subset \mathbb{C}$ denote any differentiable parametrization of the unit circle counterclockwise. For each complex differentiable function $f : \Omega \to \mathbb{C}$ defined on a neighborhood Ω of the unit circle and such that $0 \notin f(S^1)$, define the **degree** $\mathrm{Deg}\,(f, 0)$ of the function f with respect to the origin by

$$\mathrm{Deg}\,(f, 0) := n(f \circ \sigma, 0).$$

Thus, $\mathrm{Deg}\,(f, 0)$ is the winding number of the image of S^1 by f. $\square$

Exercise 79. For each non-zero integer $n \in \mathbb{Z}\backslash\{0\}$, consider the function f defined by $f(z) = z^n$; calculate $\mathrm{Deg}\,(f, 0)$.

Exercise 80. Prove that for each integer $n \in \mathbb{Z}$ the function $z \mapsto z^{-n}$ has a primitive in $\mathbb{C}^*$ if and only if $n \neq 1$.

Exercise 81. Prove that the function $z \mapsto 1/z$ has a primitive in the disc $D(1, 1)$.

Exercise 82. Prove a version of the Cauchy–Goursat theorem involving triangles instead of rectangles.

Exercise 83. Prove that Cauchy's integral representation along the unit circle $S^1(0, 1)$,

$$f(z) = \frac{1}{2\pi i} \oint_{S^1(0,1)} \frac{f(w)}{w - z}\, dw,$$

also holds for each $z \in D(0, 1)$ and each function $f : \overline{D(0, 1)} \to \mathbb{C}$ that is complex-differentiable on the open unit disc $D(0, 1)$, and continuous — but not necessarily differentiable — on the closed unit disc $\overline{D(0, 1)}$.

2.2 Complex derivatives of line integrals

The following six exercises outline another proof of the differentiability of power series.

Definition 24. A sequence (f_n) of functions $f_n : \mathcal{D} \to \mathbb{C}$ is a **uniform Cauchy sequence** on a subset $E \subseteq \mathcal{D} \subseteq \mathbb{C}$ if, and only if, for each real $\varepsilon > 0$, there exists an index $N \in \mathbb{N}$, such that $|f_m(z) - f_n(z)| < \varepsilon$ for all $m > N$ and $n > N$ and every $z \in E$.

A sequence (f_n) of functions $f_n : \mathcal{D} \to \mathbb{C}$ **converges uniformly** to a limit $f : \mathcal{D} \to \mathbb{C}$ on a subset $E \subseteq \mathcal{D} \subseteq \mathbb{C}$ if, and only if, for each real $\varepsilon > 0$, there exists an index $N \in \mathbb{N}$, such that $|f_n(z) - f(z)| < \varepsilon$ for every $n > N$ and every $z \in E$. $\square$

Exercise 84. Prove that if a sequence (f_n) of functions $f_n : \mathcal{D} \to \mathbb{C}$ is a *uniform Cauchy sequence* on a subset $E \subseteq \mathcal{D} \subseteq \mathbb{C}$, then it converges *uniformly* on E.

3 Fundamental properties of holomorphic functions

Exercise 104. For each open set $\Omega \subset \mathbb{C}$ with a connected open subset $U \Subset \Omega$ such that its closure $\overline{U}$ is compact and $\overline{U} \subset \Omega$, prove that if $f \in \mathcal{H}(\Omega)$ is not constant but $|f|$ remains constant on ∂U, then f has at least one zero in U.

Exercise 105. Prove that if $\Omega \subseteq \mathbb{C}$ and $f \in \mathcal{H}(\Omega)$, then $f^{-1}(\mathbb{R})$ is *not* a non-empty compact subset of Ω.

Exercise 106. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and that there exist positive reals A and B such that $|f(z)| \leq A|z|^n + B$ for every $z \in \mathbb{C}$. Prove that f is a polynomial.

Exercise 107. (Phragmén–Lindelöf Theorem.) Consider the domain

$$\mathcal{D} := \{z \in \mathbb{C} : \ -\pi/2 < \Re e(z) < \pi/2, \ 0 < \Im m(z)\}.$$

Suppose that $f \in \mathcal{H}(\mathcal{D}) \cap C^0(\overline{\mathcal{D}})$; thus, f is holomorphic in $\mathcal{D}$ and continuous on the closure $\overline{\mathcal{D}}$. Assume that

$$|f(z)| \leq 1, \forall z \in \partial\mathcal{D},$$

and assume that there exist reals $M > 0$ and α with $0 < \alpha < 1$ such that

$$|f(z)| \leq M \cdot \exp\left[e^{\alpha \Im m(z)}\right], \forall z \in \mathcal{D}.$$

Prove that $|f(z)| < 1$ in $\mathcal{D}$, unless f is constant. *Hint*: For each $\varepsilon > 0$ and β with $\alpha < \beta < 1$, prove that

$$\lim_{\mathrm{Im}(z)\to\infty} f(z) \exp[-\varepsilon \cdot \cos(\beta z)] = 0.$$

Determine whether the foregoing result is true for $\alpha = 1$.

3.2 Holomorphic functions

Exercise 108. This exercise shows that the existence of infinitely many real derivatives is a condition weaker than the existence of one complex derivative

(108.1) Verify that the "ramp" function $r : \mathbb{R} \to \mathbb{R}$ defined by

$$r(x) := \begin{cases} \exp(-1/x) & \text{if } 0 < x, \\ 0 & \text{if } x \leq 0, \end{cases}$$

is of class C^∞ (has derivatives of all orders at the origin). For instance, prove by induction that each derivative is the product of r and a rational function; at the origin, apply the definition of the derivative.

(108.2) Verify that the Taylor series of r at the origin converges on all of $\mathbb{R}$, but *not* to r.

Exercise 109.

(109.1) Show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) := \begin{cases} \exp\left(-z^{-4}\right) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

satisfies the Cauchy–Riemann equations on $\mathbb{C}$ but does *not* have a complex derivative at 0.

(109.2) Show that the function $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) := \begin{cases} z^5/|z|^4 & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases}$$

is continuous and satisfies the Cauchy–Riemann equations *at* 0, but does *not* have a complex derivative at 0.

Exercise 110. Prove that if a power series $\sum_{n=0}^{\infty} a_n z^n$ has a finite positive radius of convergence $R \in]0, \infty[$, and if $a_n \geq 0$ for every index n, then the holomorphic function f defined by $f(z) := \sum_{n=0}^{\infty} a_n z^n$ has a "singularity" at $z := R$, in the sense that f cannot be extended holomorphically to $D(0, R) \cup D(R, \delta)$ for any $\delta > 0$.

Exercise 111. Prove that if $R > 0$, $N \in \mathbb{N}\backslash\{0\}$, and f is holomorphic on $D(0, 2R)$ with

$$\left|f^{(N)}(0)\right| = N! R^{-N} \sup\{|f(z)| : |z| = R\},$$

then $f(z) = c \cdot z^N$ for some $c \in \mathbb{C}$.

Exercise 112. Prove that if $P : \mathbb{C} \to \mathbb{C}$ is a polynomial, if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, and if there exists a real $C > 0$ such that $|f(z)| \leq C \cdot |P(z)|$ for every $z \in \mathbb{C}$, then $f = c \cdot P$ for some $c \in \mathbb{C}$. Is there an analogous statement with P replaced by an arbitrary holomorphic function on all of $\mathbb{C}$?

Exercise 113. Prove that if $f : \overline{D(0,1)} \to \mathbb{C}$ is continuous on the closed unit disc $\overline{D(0,1)} \subset \mathbb{C}$ and holomorphic on the open unit disc $D(0,1) \subset \mathbb{C}$, and if there exists some $\alpha \in]0, 2\pi[$ such that f remains constant along the arc $A = \{e^{i\theta} : \theta \in [0, \alpha]\}$, then f is constant throughout $\overline{D(0,1)}$.

4 Theorems of Weierstrass and Montel

Exercise 125. This exercise leads to yet another proof of the differentiability of power series.

(125.1) In Weierstrass's theorem, replace "holomorphic" by "complex differentiable," and verify that the resulting proof does not depend upon power series.

(125.2) Apply the foregoing version of Weierstrass's theorem and Abel's lemma to prove the differentiability of power series.

Exercise 126. Justify the swap of limit and integral carefully to prove that for each function $f : \mathcal{D} \to \mathbb{C}$ continuous on an open set $\mathcal{D} \subseteq \mathbb{C}$, and for each differentiable curve $\gamma : [0,1] \to \mathcal{D}$, the function g defined on $\mathbb{C} \setminus \gamma([0,1])$ defined by

$$g(z) := \int_\gamma \frac{f(w)}{w - z}\, dw$$

is holomorphic, with derivatives

$$g^{(n)}(z) = n! \int_\gamma \frac{f(w)}{(w - z)^{n+1}}\, dw.$$

5 Meromorphic functions

Exercise 127. Consider a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ with Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ at the origin. Find a necessary and sufficient condition on the complex coefficients (c_n) so that $f(\mathbb{R}) \subseteq \mathbb{R}$.

Exercise 128.

(128.1) Design a multiplication algorithm for power series: With

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_{f,n} z^n, \\ g(z) &= \sum_{n=0}^{\infty} c_{g,n} z^n, \\ h(z) = f(z)g(z) &= \sum_{n=0}^{\infty} c_{h,n} z^n, \end{aligned}$$

derive an algorithm to calculate the coefficients $c_{h,n}$ of the product in terms of arithmetic operations with the coefficients $c_{f,n}$ and $c_{g,n}$. Such an algorithm may (but need not) mimic that for the multiplication of integers or of polynomials, but starting with the smallest power instead of the highest power.

(128.2) For each f and each g, identify the domain of convergence of the power series for the product $h = fg$.

(128.3) Extend the algorithm to the product of meromorphic functions.

Exercise 129.

(129.1) Design a long division algorithm for power series: With

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} c_{f,n} z^n, \\ g(z) &= \sum_{n=0}^{\infty} c_{g,n} z^n, \end{aligned}$$

$$h(z) = \frac{f(z)}{g(z)} = \sum_{n=0} c_{h,n} z^n,$$

derive an inductive algorithm to calculate the coefficients $c_{h,n}$ of the quotient in terms of arithmetic operations with the coefficients $c_{f,n}$ and $c_{g,n}$. Such an algorithm may (but need not) mimic that for the long division of integers or polynomials, but starting with the smallest power instead of the highest power.

(129.2) Identify the domain of convergence of the power series of the quotient.

(129.3) Extend the algorithm to the quotient of meromorphic functions.

Definition 27. For each $c \in \mathbb{C}$ and for all $r_1, r_2 \in \overline{\mathbb{R}}$, let $A(c, r_1, r_2) := \{z \in \mathbb{C} : r_1 < |z - c| < r_2\}$. □

Exercise 130. For each $c \in \mathbb{C}$, for all $r_1, r_2 \in \overline{\mathbb{R}}$, and for each $f \in \mathcal{H}[A(c, r_1, r_2)]$, prove that there exist $f_1 \in \mathcal{H}[A(c, r_1, \infty)]$ and $f_2 \in \mathcal{H}[D(c, r_2)]$ such that $f = f_1 - f_2$. Also prove that there exist only one pair of such functions f_1 and f_2 with $\lim_{|z|\to\infty} |f_1(z)| = 0$.

Exercise 131. Prove that if f is entire and not constant, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

2 The residue theorem

The following exercise provides a "pertubation theorem" for equations.

Exercise 202. Prove that if a holomorphic function $f : \Omega \to \mathbb{C}$ has a zero of order m at a point $c \in \Omega$, then there exist positive radii r and R such that for each $w \in D(0, R)$ the equation $f(z) = w$ has m solutions, counted with multiplicities, in $D(c, r)$.

Exercise 203. Prove that for each w in the unit disc $D(0, 1)$, the equation $z^5(z - 2) = w$ has exactly five solutions in $D(0, 1)$, counted with multiplicities.

Exercise 204. Let f be a meromorphic function on a simply connected open set $\Omega \subseteq \mathbb{C}$. Prove that f has a primitive that is meromorphic on Ω if and only if at each pole of f the residue of f is zero.

Exercise 205. Show that there is *no* holomorphic function $f \in \mathcal{H}(\mathbb{C} \setminus \{1, -1\})$ with

$$f'(z) = \frac{1}{z^2 - 1}.$$

Determine whether the same results holds for $\mathbb{C} \setminus \overline{D(0, 1)} = \{z \in \mathbb{C} : 1 < |z|\}$.

Exercise 206. This exercise provides a quantitative version of the statement that if $f'(0) \neq 0$ then f is injective in a neighborhood of 0. Specifically, consider a function $f \in \mathcal{H}[D(0, 1)]$ with a MacLaurin series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

such that

$$|c_1| > \sum_{n=2}^{\infty} n|c_n|.$$

Prove that f is injective in the open unit disc $D(0, 1)$.

The following exercises pertain to doubly periodic meromorphic functions, with the following notation.

Let $\omega_1 \in \mathbb{C}$ and $\omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$.

Let $L := \{n_1\omega_1 + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$.

For each $z_0 \in \mathbb{C}$ let $P(z_0) := \{z_0 + t_1\omega_1 + t_2\omega_2 : 0 \leq t_1 \leq 1, 0 \leq t_2 \leq 1\}$.

Definition 35. A function f is **doubly periodic** if and only if f is a meromorphic function on $\mathbb{C}$ such that $f(z + \ell) = f(z)$ for every $\ell \in L$ and every $z \in \mathbb{C}$ where f is holomorphic. □

Exercise 207. Prove that every *entire* doubly periodic function is constant.

Exercise 208. Prove that if $\partial P(z_0)$ does not contain any pole of f then

$$\oint_{\partial P(z_0)} f(z)\,dz = 0.$$

Exercise 209. Prove that if $\partial P(z_0)$ does not contain any zero and any pole of f, then the interior of $P(z_0)$ contains equally many zeroes as poles of f, all counted with their multiplicities.

Exercise 210. If $\partial P(z_0)$ does not contain any zero and any pole of f, let $z_1, \ldots, z_n$ be the zeroes of f and let $w_1, \ldots, w_n$ be the poles of f, all counted with their multiplicities, in the interior of $P(z_0)$. Prove that $\sum_{k=1}^{n}(z_k - w_k) \in L$. *Hint:* Consider

$$\oint_{\partial P(z_0)} \frac{zf'(z)}{f(z)}\,dz.$$

Picard's Theorem

Exercise 220. Prove that if f and g are entire and $e^f + e^g = 1$, then f and g are constant.

1 Partitions of unity

These exercises focus on details of the proof of the existence of partitions of unity.

Exercise 224. This exercise merely verifies that $C_0^\infty \neq \varnothing$.

(224.1) Verify that the "ramp" function $r : \mathbb{R} \to \mathbb{R}$ defined by

$$r(x) := \begin{cases} \exp(-1/x) & \text{if } 0 < x, \\ 0 & \text{if } x \leq 0, \end{cases}$$

is of class C^∞ (has derivatives of all orders at the origin).

(224.2) Verify that the "bump" function $b : \mathbb{R} \to \mathbb{R}$ defined by

$$b(x) := r(1+x)r(1-x)$$

is of class C^∞, positive in $]-1, 1[$, and has compact support $[-1, 1]$. In particular, $0 < \int_{\mathbb{R}} b < \infty$. Sketch b.

(224.3) Verify that the function $\rho : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\rho(\vec{\mathbf{x}}) := b\left(\frac{\|\vec{\mathbf{x}} - \vec{\mathbf{w}}\|_2^2}{\delta^2}\right)$$

is of class C^∞, non-negative, positive in the interior of the ball $B(\vec{\mathbf{w}}, \delta)$, and has a compact support $\overline{B(\vec{\mathbf{w}}, \delta)}$.

Definition 36. A **distance,** or **metric,** on a set X is a function $d : (X \times X) \to \mathbb{R}_+$ such that for all $u, v, w \in X$,

$$\begin{aligned} d(u,v) &= 0 && \text{if and only if, } u = v; \\ d(u,v) &= d(v,u); \\ d(u,w) &\leq d(u,v) + d(v,w). \end{aligned}$$

For each $x \in X$, and for each non-empty subset $Z \subseteq X$, define a function $d^b(\ ,Z) : X \to \mathbb{R}_+$ by

$$d^b(x, Z) := \inf\{d(x,z) : z \in Z\}.$$

□

Exercise 225.

(225.1) Prove that $|d^b(u, Z) - d^b(v, Z)| \leq d(u, v)$.

(225.2) Prove that if $(X, \mathcal{T})$ is a topological space and if d is continuous on the product space, then d^b is continuous on X.

Exercise 226. Prove that for each open set $U \subseteq \mathbb{R}^N$, there exists a sequence $(K_j)_{j\in\mathbb{N}}$ of compacta $K_j \subset \mathbb{R}^N$ such that

$$\begin{aligned} U &= \bigcup_{j\in\mathbb{N}} K_j, \\ K_j &\subset (K_{j+1})^\circ \end{aligned}$$

(where the superscript $^\circ$ denotes the topological interior).

Exercise 227. For each collection $\mathcal{U}$ of open subsets of $\mathbb{R}^N$, let $U := \bigcup \mathcal{U}$, and let $(K_j)_{j\in\mathbb{N}}$ be a sequence of compact sets as in Exercise 226.

(227.1) Prove that for each $j > 1$ and for each $\varepsilon > 0$, there exists a cover of $K_j \setminus (K_{j-1})^\circ$ by finitely many open balls with radii less than ε and whose closures lie entirely within $U \setminus K_{j-2}$.

(227.2) Deduce that $\mathcal{U}$ admits a locally finite refinement with such open balls.

(227.3) With smooth functions supported in the closures of such balls, conclude that each open cover $\mathcal{U}$ admits a partition of unity.

2.3 Inhomogeneous Cauchy–Riemann equations

Exercise 241. For each non-empty open set $\Omega \subseteq \mathbb{C}$ and for each continuously differentiable function $\varphi : \Omega \to \mathbb{C}$ such that the partial differential equation $\partial_{\bar{z}} u = \varphi$ in Ω has at least one continuously differentiable solution $u : U \to \mathbb{C}$, identify *all* the continuously differentiable solutions v of $\partial_{\bar{z}} v = \varphi$ in Ω in terms of the solution u and other types of functions.

Exercise 242. Show that for some compactly supported differentiable function φ, *none* of the solutions u of $\partial_{\bar{z}} u = \varphi$ has a compact support.

Exercise 243. Prove that for each theorem that guarantees the existence of a solution u for the "$\bar{\partial}$" equation $\partial_{\bar{z}} u = \varphi$, there is a corresponding theorem that guarantees the existence of a solution v for the "∂" equation $\partial_z v = \varphi$.

Exercise 244. For each non-empty open set $\Omega \subseteq \mathbb{C}$, for each continuously differentiable function $\varphi : \Omega \to \mathbb{C}$, and for each relatively compact non-empty open subset $U \Subset \Omega$, prove that the partial differential equation $\Delta u = \varphi$ in U has a solution $u : U \to \mathbb{C}$ that is continuous on the closure $\overline{U}$.

Exercise 245. For each $\varphi \in C_0^\infty(\mathbb{C})$ with $\varphi \geq 0$ on $\mathbb{C}$, prove that if there exists $u \in C_0^\infty(\mathbb{C})$ such that $\partial u/\partial \bar{z} = \varphi$, then $\varphi \equiv 0$.

Exercise 246. For every $\varphi \in C_0^\infty(\Omega, \mathbb{C})$ and every $n \in \mathbb{N}$, prove that

$$\iint_{\mathbb{C}} \frac{\partial \varphi}{\partial \bar{z}} \cdot z^n \cdot dx \wedge dy = 0.$$

More generally, prove that for each $f \in \mathcal{H}(\Omega)$,

$$\iint_{\Omega} \frac{\partial \varphi}{\partial \bar{z}} \cdot f(z) \cdot dx \wedge dy = 0.$$

Exercise 247. For each $\psi \in C_0^\infty(\mathbb{C})$ such that $\int\int_{\mathbb{C}} \psi(z) \cdot z^n \, dx \, dy = 0$ for every integer $n \geq 0$, prove that there exists $u \in C_0^\infty(\mathbb{C})$ with $\partial u/\partial \bar{z} = \psi$.

Exercise 248. Consider $\varphi \in C_0^\infty(\mathbb{C})$ with compact support $K := \text{support}\,(\varphi)$.

(248.1) Prove that if there exists $u \in C_0^\infty(\mathbb{C})$ with $\partial u/\partial \bar{z} = \varphi$, then $\text{support}\,(u) \subseteq \widehat{K}_{\mathbb{C}}$

(248.2) Prove that there exists at most one $u \in C_0^\infty(\mathbb{C})$ with $\partial u/\partial \bar{z} = \varphi$.

Exercise 249. Construct a connected open set $\Omega \subset \mathbb{C}$ and a function $\varphi \in C_0^\infty(\Omega)$ such that the equation $\partial u/\partial \bar{z} = \varphi$ has a solution $u \in C_0^\infty(\mathbb{C})$ but no solution $u \in$

1 The Mittag-Leffler theorem

The following exercise verifies a step in the proof of Theorem 1 in Section 1 of Chapter 6.

Exercise 279. With the notation as in the text, prove that for each open subset $\Omega \subseteq \mathbb{C}$ and for all compact subsets K and L of Ω such that

$$K \subset L^{\circ} \subset \Omega,$$

the following inclusion holds:

$$\widehat{K}_{\Omega} \subset \left(\widehat{L}_{\Omega}\right)^{\circ}.$$

Exercise 280. Let $\{D_k : k \in I\}$ be an open cover of an open set $\Omega \subseteq \mathbb{C}$ by open discs. For each index $k \in I$, let $h_k \not\equiv 0$ be a meromorphic function on D_k, and assume that for all indices $k, \ell \in I$, the function $g_{k,\ell} := h_k/h_\ell$ is holomorphic on $D_k \cap D_\ell$. Prove that for all indices $k, \ell \in I$ there exist holomorphic functions f_k on D_k without any zero and such that $f_k = g_{k,\ell} f_\ell$ on $D_k \cap D_\ell$.

Exercise 281. For each *bounded* connected open set $\Omega \subset \mathbb{C}$ and each *bounded* function $\varphi \in C^{\infty}(\Omega, \mathbb{C})$, not necessarily with compact support, prove that there exists a *bounded* function $u \in C^{\infty}(\Omega, \mathbb{C})$ such that $\partial u/\partial \overline{z} = \varphi$ on Ω.

Exercise 282. Let R_1 and R_2 be rectangles in $\mathbb{C}$ whose union $R_1 \cup R_2$ is also a rectangle. Suppose that $f : (R_1 \cap R_2) \to \mathbb{C}$ is a *bounded* holomorphic function on their intersection. Prove that there exist *bounded* holomorphic functions $f_1 \in \mathcal{H}(R_1)$ and $f_2 \in \mathcal{H}(R_2)$ such that $f = f_1 - f_2$ on $R_1 \cap R_2$.

1 Analytic automorphisms of the disc and of the annulus

Exercise 293. Determine the types of fractional linear transformations that map the upper half plane onto the unit disc, with the real axis onto the unit circle.

Exercise 294. Determine the types of fractional linear transformations that map the upper half plane onto the upper half plane.

Exercise 295.

(295.1) Prove that every analytic automorphism of the unit disc that fixes two distinct points is the identity.

(295.2) Let S be a finite subset of the open unit disc $D := D(0, 1)$, and let $\Omega := D \setminus S$. Prove that each analytic automorphism of Ω is the restriction to Ω of an analytic automorphism of D that permutes the points of S.

(295.3) For each real r such that $0 < |r| < 1$, prove that there exists at most one real s with $0 < s < 1$ for which $\Omega := D \setminus \{0, r, s\}$ admits an analytic automorphism different from the identity.

Definition 45. A continuous function $f : X \to Y$ between locally compact topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ is **proper** if and only if for each compact subset $L \subseteq Y$ the preimage $f^{-1}(L) \subseteq X$ is compact. □

Exercise 296. Prove that an entire function is proper if and only if it is a non-constant polynomial.

Exercise 297. Prove that there is *no* proper holomorphic map from the open unit disc into the complex plane.

Exercise 298.

(298.1) Consider $f \in \mathcal{H}[D(0, R)]$ with $f(0) \neq 0$ and for which there exists a real $M > 0$ such that $|f| < M$. Assume that f has n zeroes $c_1, \ldots, c_n \in D(0, R)$ and that $c_1, \ldots, c_n \in D(0, r)$ with $0 < r < R$. Prove that

$$|f(0)| \leq M \cdot \left(\frac{r}{R}\right)^n.$$

Hint: Consider

$$f(z) \cdot \prod^{n} \frac{R^2 - \overline{c_k} z}{R \cdot (z - c_k)}.$$

(298.2) Consider a *entire* function f for which there exist real constants $C > 0$ and $\rho > 0$ such that

$$|f(z)| \leq C \cdot \exp\left(|z|^{\rho}\right)$$

for every $z \in \mathbb{C}$. Prove that if $f(\log(n)) = 0$ for every integer $n \geq 3$, then $f \equiv 0$.

Exercise 299. Consider a *bounded* connected open set $\Omega \subset \mathbb{C}$ and a holomorphic map $f : \Omega \to \Omega$. Assume that there exists a point $c \in \Omega$ where $f(c) = c$ and $f'(c) = 1$.

(299.1) Prove that f is the identity. *Hint*: consider the n th iterate $f^{\circ n}$ of f, and calculate the first non-zero coefficient of the Taylor series at c of $z \mapsto f^{\circ n}(z) - z$.

(299.2) Determine whether the foregoing result holds without the hypothesis that Ω be bounded.

Exercise 300. Consider a *bounded* connected open set $\Omega \subset \mathbb{C}$ with any point $c \in \Omega$. Let G be the group of analytic automorphisms of Ω that leave c fixed. Prove that G is compact: every sequence of elements of G has a subsequence converging uniformly on compact subsets of Ω to an element of G.

Definition 46. For each open set $\Omega \subseteq \mathbb{C}$ let $H^{\infty}(\Omega)$ denote the space of *bounded* holomorphic functions on Ω. For each $f \in H^{\infty}(\Omega)$ let

$$\|f\|_{\Omega,\infty} := \sup\{|f(z)| : z \in \Omega\}.$$ □

Exercise 301.

(301.1) Consider a function $f \in H^{\infty}[D(c, R)]$; thus $f \in \mathcal{H}[D(c, R)]$ with $|f| \leq M$. Also let r be a real with $0 < r < R$. Prove that if $f^{(n)}(c) = 0$ for every $n \in \{0, \ldots, m\}$, then

$$|f(z)| \leq M \cdot \left(\frac{r}{R}\right)^m$$

for every $z \in D(c, r)$.

(301.2) For each *connected* open set $\Omega \subseteq \mathbb{C}$, for each compact subset $K \subset \Omega$, and for each real $\varepsilon > 0$, prove that there exists a linear subspace $V \subset H^{\infty}(\Omega)$ with *finite* codimension such that for every $z \in K$ and every $f \in V$

$$|f(z)| \leq \varepsilon \cdot \|f\|_{\Omega,\infty}$$

for every $z \in K$ and every $f \in V$.

Definition 47. An **analytic automorphism** of an open set $\Omega \subseteq \mathbb{C}$ is a bijective holomorphic function $f : \Omega \to \Omega$ (it then follows that $f^{-1} : \Omega \to \Omega$ is also holomorphic).

The group (with composition of functions) of all analytic automorphisms of Ω is denoted by $\mathrm{Aut}(\Omega)$.

For each $c \in \Omega$, the subgroup of all analytic automorphisms of Ω that fix c, so that $f(c) = c$, is denoted by $\mathrm{Aut}_c(\Omega)$. □

Exercise 302. For each $c \in D(0, 1)$ and each real θ define $\varphi_{c,\theta} \in \mathrm{Aut}[D(0, 1)]$ by

$$\varphi_{c,\theta}(z) := e^{i\theta} \cdot \frac{z - c}{1 - \bar{c}z}.$$

Consider a subgroup $G \subseteq \mathrm{Aut}[D(0, 1)]$ such that $G \supseteq \mathrm{Aut}_0[D(0, 1)]$.

(302.1) Prove that if $\varphi_{c_0,\theta_0} \in G$, then $\varphi_{c,\theta} \in G$ for every c such that $|c| = |c_0|$ and every $\theta \in \mathbb{R}$.

(302.2) Deduce that if $G \neq \mathrm{Aut}_0[D(0, 1)]$ then $G = \mathrm{Aut}[D(0, 1)]$. *Hint*: calculate $\varphi_{c,\theta} \circ \varphi_{c,\theta}$ for $c > 0$.

Exercise 303. Consider a *convex* open set $\Omega \subset \mathbb{C}$ and a biholomorphic map $f : D(0,1) \to \Omega$. For each real r with $0 < r < 1$ let $\Omega_r := f[D(0,r)]$. Prove that Ω_r is convex. *Hint*: assume that $f(0) = 0$, and for all $p, q \in D(0,1)$ with $|p| \leq |q| < 1$ and $q \neq 0$ consider $f^{-1}\left[(1-t) \cdot f(pq^{-1}z) + t \cdot f(z)\right]$.

Exercise 304. Consider open sets $\Omega_1 \subseteq \mathbb{C}$ and $\Omega_2 \subseteq \mathbb{C}$ with Ω_2 connected, and let $f : \Omega_1 \to \Omega_2$ be a *proper* holomorphic map. Prove that the number of solutions $z \in \Omega_1$ of the equation $f(z) = w$, in other words, the cardinality of $f^{-1}(\{w\})$ counted with multiplicities, is constant on Ω_2.