1. Differenial Forms and Integration

1.1. One variable. We start by recalling two important results from one variable analysis.

**Theorem 1.1. (Fundamental Theorem of Analysis)** Let \( f \in C([a,b]) \). Then the function

\[
g(x) := \int_{a}^{x} f(t)dt
\]

satisfies \( g'(x) = f(x) \). Moreover, if \( G(x) \) is any function differentiable on \([a,b]\) such that \( F'(x) = f(x) \), we have that

\[
\int_{a}^{b} f(t)dt = F(b) - F(a).
\]

**Theorem 1.2. (Change of variable formula in one variable)** Let \( h: [a_1,b_1] \to [a_2,b_2] \) be a differentiable function such that \( h(a_1) = a_2, h(b_1) = b_2 \) and \( h'(x) \neq 0 \) for all \( x \in [a_1,b_1] \). Let \( f \in C[a_2,b_2] \). Then

\[
\int_{a_2}^{b_2} f(t)dt = \int_{a_1}^{b_1} f(h(t))h'(t)dt.
\]

1.2. Two variables. In one variable, the derivative of a function at a point is a measure of infinitesimal change. In two (or more) variables one has to specify a direction in which one wants to measure change. Define linear maps \( dx: \mathbb{R}^2 \to \mathbb{R} \) and \( dy: \mathbb{R}^2 \to \mathbb{R} \) by

\[
dx(v) = dx((v_1,v_2)) := v_1 \quad \text{and} \quad dy(v) = dx((v_1,v_2)) := v_2.
\]

**Definition 1.3.** Let \( f \in C^1(U) \) for an open set \( U \subset \mathbb{R}^2 \). The differential of \( f \) is defined as

\[
df(x,y) := \frac{\partial f}{\partial x}(x,y)dx + \frac{\partial f}{\partial y}(x,y)dy.
\]

**Example 1.4.** Fix a point \((x,y) \in \mathbb{R}^2 \) and let \( f \) be a \( C^1 \)-smooth function defined near \((x,y) \). Let \( \gamma: (-\epsilon, \epsilon) \to \mathbb{R}^2 \) be a \( C^1 \)-smooth curve with \( \gamma(0) = (x,y) \). Then \( g(t) := f(\gamma(t)) \) is a function of one variable. Moreover, by the chain rule,

\[
g'(0) = \frac{\partial f}{\partial x}(x,y) \cdot \gamma'_1(0) + \frac{\partial f}{\partial y}(x,y) \cdot \gamma'_2(0) = df(x,y)(\gamma'(0)).
\]

**Definition 1.5.** A continuous differential 1-form \( \omega \) on an open set \( U \subset \mathbb{R}^2 \) is an expression

\[
\omega(x,y) = P(x,y)dx + Q(x,y)dy,
\]

**Definition 1.6.** Let \( \gamma: [0,1] \to \mathbb{R}^2 \) be a \( C^1 \)-smooth map such that \( \gamma'(t) \neq 0 \) for all \( t \), and let \( \omega(x,y) \) be a differential 1-form defined on an open set
containing \( \gamma([0,1]) \). We set

\[
\int_\gamma \omega := \int_0^1 \omega(\gamma(t))(\gamma'(t))dt. \tag{1.8}
\]

**Proposition 1.7.** Let \( \gamma \subset \mathbb{R}^2 \) be an oriented \( C^1 \)-smooth curve. Then \( \int_\gamma \omega \) is independent of parametrizations respecting a given orientation.

**Proof.** Set \( h(t) = \gamma_1^{-1}(\gamma_2(t)) \), such that \( \gamma_2(t) = \gamma_1(h(t)) \). Using Theorem 1.2 we get that

\[
\int_0^1 \omega(\gamma_2(t))(\gamma_2'(t))dt = \int_0^1 \omega(\gamma_1(h(t)))(\gamma_1'(h(t))h'(t))dt \\
= \int_0^1 \omega(\gamma_1(t))(\gamma_1'(t))dt.
\]

\( \square \)

We immediately get the analogue of (1.2):

**Theorem 1.8.** Let \( \gamma \) be a \( C^1 \)-smooth oriented curve with end points \( a \) and \( b \), and assume that \( f \) is a \( C^1 \)-smooth function defined on an open set containing \( \gamma \). Then

\[
\int_\gamma df = f(b) - f(a). \tag{1.9}
\]

**Proof.**

\[
\int_\gamma df = \int_0^1 df(\gamma(t))(\gamma'(t))dt = \int_0^1 \frac{d}{dt}(f(\gamma(t)))dt \\
= f(\gamma(1)) - f(\gamma(0)) = f(b) - f(a).
\]

\( \square \)

Let us now consider the analogue of (1.1), i.e., given a 1-form \( \omega \) on an open set \( U \), does there exist a function \( f \) with \( df = \omega \)? This turns out to be more subtle. We see immediately that (1.9) gives a necessary condition for such a function to exist: for any two points \( a, b \in U \) and any two curves \( \gamma_j, j = 1, 2 \), connecting \( a \) and \( b \), we need to have that \( \int_{\gamma_1} \omega = \int_{\gamma_2} \omega \). is the same. In particular, for any closed loop, the integral needs to be zero. It turns out that this condition is also sufficient. Pick any point \( a \in U \). Then for any point \( b \in U \) let \( \gamma \) be any curve connecting \( a \) and \( b \) and set \( f(b) := \int_\gamma \omega \). The proof of the fact that \( df = \omega \) is the same as that of Theorem 1.14 below.

We now want to consider another analogue of (1.2). For this we need to define the "derivative" of a 1-form. If \( \alpha, \beta : \mathbb{R}^2 \to \mathbb{R} \) are linear maps, we define first a bi-linear map \( \alpha \land \beta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) by

\[
\alpha \land \beta(u, v) := \alpha(u) \cdot \beta(v) - \alpha(v) \cdot \beta(u). \tag{1.10}
\]
Example 1.9. $dx \wedge dy(u,v) = u_1 \cdot v_2 - v_1 \cdot u_2$. So this is simply the determinant of the matrix whose rows are $u$ and $v$. So it is the area of the parallelepiped spanned by the two vectors.

The following is easily verified.

1. $\alpha \wedge \beta = -\beta \wedge \alpha$,
2. $\alpha \wedge (\beta_1 + \beta_2) = \alpha \wedge \beta_1 + \alpha \wedge \beta_2$,
3. $\alpha \wedge \alpha = 0$.

Definition 1.10. Let $\omega(x,y) = P(x,y)dx + Q(x,y)dy$ be a $C^1$-smooth differential 1-form defined on an open set $U \subset \mathbb{R}^2$. We define

\begin{align*}
(1.11)\quad d\omega := dP \wedge dx + dQ \wedge dy.
\end{align*}

We see that

\begin{align*}
d\omega &= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy\right) \wedge dy \\
&= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.
\end{align*}

Definition 1.11. Let $U \subset \mathbb{R}^2$ be an open set. A differential 2-form on $U$ is an expression

\begin{align*}
(1.12)\quad \Omega(x,y) = R(x,y)dx \wedge dy,
\end{align*}

where $R$ is a function on $U$.

Definition 1.12. Let $U \subset \mathbb{R}^2$ be an open set and let $\Omega$ be a differential 2-form on $U$. We define

\begin{align*}
(1.13)\quad \int \int_U \Omega := \int \int_U R(x,y)dxdy.
\end{align*}

We now get the other analogue of (1.2) from Green’s Theorem: Let $U \subset \mathbb{R}^2$ be a domain with piecewise smooth boundary $bU$, and let $\omega$ be a $C^1$-smooth differential 1-form defined in an open neighbourhood of $U$. Then

\begin{align*}
(1.14)\quad \int_{bU} \omega = \int \int_U d\omega.
\end{align*}

Proposition 1.13. Let $f \in C^2(U)$ for an open set $U \subset \mathbb{R}^2$. Then $df = 0$.

Proof. This follows from the equality of mixed partials. $\square$

Theorem 1.14. Let $U \subset \mathbb{R}^2$ be a star-shaped domain, and suppose $\omega$ is a 1-form of class $C^1$ on $U$. Suppose further that $d\omega = 0$. Then there exists $f \in C^1(U)$ such that $\omega = df$. 
Proof. We may assume that $U$ is star-shaped with respect to origin. For each point $(x, y) \in U$ we let $l_{x,y}$ denote the straight line segment between 0 and $(x, y)$. Set

\[(1.15) \quad f(x, y) := \int_{l_{x,y}} \omega = \int_{l_{x,y}} P \, dx + Q \, dy.\]

Then $f$ is a well defined continuous function on $U$. Pick a point $a = (x_0, y_0) \in U$. We will show that $\frac{\partial f}{\partial x}$ exists at $a$ and is equal to $P(a)$.

Choose $\epsilon > 0$ such that the line segment $I_\epsilon := \{(x, y) : |x - x_0| \leq \epsilon\}$ is contained in $U$. Then by Green’s theorem, for all $x$ with $|x - x_0| \leq \epsilon$ we have that

\[(1.16) \quad f(x, y_0) = \int_{l_{x_0-\epsilon, y_0}} \omega + \int_{x_0-\epsilon}^{x} P(t, y_0)dt.\]

It follows that $\frac{df}{dx}(x_0, y_0) = P(x_0, y_0)$. The $y$-derivative is completely similar. $\square$

**Example 1.15.** Consider $\omega(x, y) = \frac{xdy}{x^2+y^2} - \frac{ydx}{x^2+y^2}$, a differential 1-form on $\mathbb{R}^2 \setminus \{0\}$. Then

\[d\omega(x, y) = \left[(x^2 + y^2) - 2x^2\right]dx \wedge dy - \left[(x^2 + y^2) - 2y^2\right]dy \wedge dx = \frac{[(x^2 + y^2) - 2x^2 + (x^2 + y^2) - 2y^2]dx \wedge dy}{x^2 + y^2} = 0.\]

On the other hand, consider the curve $\gamma(t) = (\cos t, \sin t), t \in [0, 2\pi]$. Then

\[\int_{\gamma} \omega = \int_{0}^{2\pi} \cos^2 t - (-\sin^2 t)dt = 2\pi.\]

So there is no function $f$ such that $df = \omega$ even though $d\omega = 0$, because in that case the integral should have been zero.

1.3. **Transforming Tangents and Differential Forms.** Let $U$ and $V$ be open sets in $\mathbb{R}^2$, and suppose there exists a $C^1$-smooth map $F : U \to V$ with a $C^1$-smooth inverse $G : V \to U$. Then $F$ can be used to push forward curves from $U$ to $V$, and pull back functions from $V$ to $U$. More precisely, if $\gamma : (-\epsilon, \epsilon) \to U$ is a curve, then $F \circ \gamma : (-\epsilon, \epsilon) \to V$ is a curve, denoted by $F_* \gamma$, and if $g$ is a function on $V$, then $g \circ F$ is a function on $U$. This function will be denoted by $F^* g$.

Next we see that

\[(1.17) \quad \frac{d}{dt} F(\gamma(t))(0) = DF(\gamma(0))(\gamma'(0)).\]

So the Jacobian $DF$ at $\gamma(0)$ maps the tangent to $\gamma$ at $\gamma(0)$ to the tangent to $F \circ \gamma$ at $F(\gamma(0))$. So for any point $p \in U$ we define a map $F_* : T_p U \to T_{F(p)} V$ by $f_* v := DF(p)(v)$. 
Next consider \( F^*g = g(F(x, y)) \). We see that
\[
d(F^*g)(p)(v) = [dg(F(p))DF(p)](v) = dg(F(p))(DF(p)(v))
\]
\[
= dg(F(p))F_*v.
\]
This suggests that we for any 1-form \( \omega \) on \( V \) define
\[
(1.18) \quad F^*\omega(p)(v) := \omega(F(p))(F_*v),
\]
and get the nice formula
\[
(1.19) \quad dF^*g = F^*dg.
\]

**Proposition 1.16.** Let \( \gamma : [0, 1] \rightarrow V \) be a \( C^1 \)-smooth curve, and let \( \omega \) be a differentiable 1-form on \( V \). Then
\[
(1.20) \quad \int_{\gamma} \omega = \int_{F^{-1}\gamma} F^*\omega.
\]

**Proof.**
\[
\int_{\gamma} \omega = \int_{0}^{1} \omega(\gamma(t))(\gamma'(t)) = \int_{0}^{1} \omega(F(F^{-1}(\gamma(t))))(F_*((F^{-1}_*(\gamma(t)))))
\]
\[
= \int_{0}^{1} F^*\omega(F^{-1}(\gamma(t)))(F_*^{-1}(\gamma'(t)))
\]
\[
= \int_{F^{-1}\gamma} F^*\omega.
\]
\[\square\]

Next, suppose that \( \alpha \wedge \beta \) is a differential 2-form on \( V \). We define
\[
(1.21) \quad F^*(\alpha \wedge \beta) := F^*\alpha \wedge F^*\beta.
\]
So we get that \( F^*(\alpha \wedge \beta)(p)(u, v) = \alpha \wedge \beta(F(p))(F_*u, F_*v) \). In particular
\[
(1.22) \quad F^*(dx \wedge dy) = \det(F'(p))dx \wedge dy.
\]
Moreover, we have the formula
\[
(1.23) \quad F^*(d\omega) = dF^*\omega.
\]
To see this, write \( \omega = f dx + gdy \). We get that
\[
F^*d\omega = F^*(df \wedge dx + dg \wedge dy)
\]
\[
= F^*df \wedge F^*dx + F^*dg \wedge F^*dy
\]
\[
= dF^*f \wedge F^*dx + dF^*g \wedge F^*dy
\]
\[
= d(F^*f \cdot F^*dx + F^*g \cdot F^*dy)
\]
\[
= dF^*\omega.
\]
1.4. **Complex Valued Differential Forms.** Next we allow complex coefficients, and consider functions and forms

(1.24) \( f(z), \omega(z) = g(z)dx + h(z)dy, \) and \( \Omega(z) = R(z)dx \wedge dy, \)

where \( f, g, h \) and \( R \) are complex valued, and extend everything above by linearity. It becomes convenient to introduce:

(1.25) \[ dz := dx + idy \quad \text{and} \quad d\bar{z} := dx - idy. \]

Then we get

\[
\omega(z) = \frac{1}{2} g(z)(dz + d\bar{z}) - \frac{i}{2} h(z)(dz - d\bar{z})
= \frac{1}{2}(g(z) - ih(z))dz + \frac{1}{2}(g(z) + ih(z))d\bar{z}.
\]

If \( f \) is a \( C^1 \)-smooth function on \( U \) we get that

(1.26) \[ df(z) = \frac{\partial f}{\partial z}(z)dz + \frac{\partial f}{\partial \bar{z}}(z)d\bar{z} =: \partial f(z) + \bar{\partial} f(z). \]

In general, we say that a 1-form \( \omega(z) = P(z)dz \) is of type \((1, 0)\), and a 1-form \( \omega(z) = Q(z)d\bar{z} \) is of type \((0, 1)\).

Now consider a \( C^1 \)-smooth map \( f(x, y) = (u(x, y), v(x, y)) : \mathbb{R}^2 \to \mathbb{R}^2 \)
defined near the origin Then

(1.27) \[ f(x, y) = f(0) + (du(0)(x, y), dv(0)(x, y)) + o(||(x, y)||). \]

Writing this on complex form, \( i.e., \) writing \( f = u + iv \) and \( z = x + iy \), we get

\[
f(z) = f(0) + du(0)(x, y) + idv(0)(x, y) + o(|z|)
= f(0) + \frac{\partial u}{\partial z}(0) \cdot (x + iy) + \frac{\partial u}{\partial \bar{z}}(0) \cdot (x - iy)
+ i(\frac{\partial v}{\partial z}(0) \cdot (x + iy) + \frac{\partial v}{\partial \bar{z}}(0)) \cdot (x - iy)) + o(|z|)
= f(0) + \frac{\partial f}{\partial z}(0) \cdot z + \frac{\partial f}{\partial \bar{z}}(0) \cdot \bar{z} + o(|z|).
\]

If we identify a vector \((\alpha, \beta)\) with the complex number \( \lambda = \alpha + i\beta \), we note that \( dz(\alpha, \beta) = \lambda \) and \( d\bar{z}(\alpha, \beta) = \bar{\lambda} \), so we adopt the convention that \( dz(\lambda) = \lambda \) and \( d\bar{z}(\lambda) = \bar{\lambda} \). With this convention we get that

(1.28) \[ f(z) = f(0) + df(0)(z) + o(|z|) = f(0) + \partial f(0)(z) + \bar{\partial} f(0)(z) + o(|z|). \]

**References**