

MAT4800

Mandatory assignment 1 of 1

Submission deadline

Thursday 31st OCTOBER 2024, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with $\text{\LaTeX}$). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Guidelines for this assignment

This mandatory assignment contains standard material/results. This means that if you find it difficult to prove the results yourself, you may find help in the literature. It you need to consult the literature that is fine, but if so, include references, and make sure you present the proofs in your own way.

Working on the results in the assignment will be useful preparation for later in the course.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Let $\Omega \subseteq \mathbb{C}$ be a domain. We let $\mathcal{E}(\Omega)$ denote the space of smooth functions on Ω .

Recall that a smooth 1-form ω on Ω is on the form

$$\omega(z) = g(z)dx + h(z)dy \quad (1)$$

where $g, h \in \mathcal{E}(\Omega)$. With the notation $dz = dx + idy, d\bar{z} = dx - idy$ any such ω may be written uniquely

$$\omega = \tilde{g}(z)dz + \tilde{h}(z)d\bar{z}. \quad (2)$$

For $f \in \mathcal{E}(\Omega)$ we define

$$df(z) = f_x(z)dx + f_y(z)dy = f_z(z)dz + f_{\bar{z}}(z)d\bar{z}.$$

We say that ω as in (1) is *closed* if $h_x(z) - g_y(z) = 0$ for all $z \in \Omega$. This is equivalent to $\tilde{h}_z(z) - \tilde{g}_{\bar{z}}(z) = 0$ for all $z \in \Omega$.

Recall that for $r > 0$ and $a \in \mathbb{C}$, we have that

$$\Delta_r(a) = \{z \in \mathbb{C} : |z - a| < r\}$$

Problem 1.

- (i) Prove that if $\omega \in \mathcal{E}^1(\Delta_r(a))$ is closed, there exists an $f \in \mathcal{E}^1(\Delta_r(a))$ such that $\omega = df$.

(Hint: In the unit disk Δ , define $f(z) = \int_{l_z} \omega$, where l_z is the straight line segment between 0 and z . Find an application of Stoke's Theorem to prove the result (we remark that Stoke's Theorem is valid for domains with piecewise smooth boundary).)

- (ii) Conclude that if $g \in \mathcal{O}(\Delta_r(a))$ there exists $f \in \mathcal{O}(\Delta_r(a))$ such that $f' = g$.

Solution:

(i) We follow the hint (proving this for a general disk $\Delta_r(a)$ amounts to the same thing). Note that f is continuous. We write

$$\omega(z) = h(z)dx + g(z)dy$$

Fix a point $z_0 = x_0 + iy_0 \in \Delta$; we need to prove that the partial derivatives of f exists at z_0 and that $f_x(z_0) = h(z_0), f_y(z_0) = g(z_0)$. Assume for simplicity that $x_0, y_0 > 0$.

Let l be the straight line segment between 0 and $(0, y_0)$, and for $x > 0$ let l_x be the straight segment between $(0, y_0)$ and (x, y_0) . By Stokes theorem we have for any $x > 0$ that

$$f(x, y_0) = \int_{l_1} \omega + \int_{l_x} \omega$$

The first integral is a constant that does not depend on x - we now want to differentiate with respect to x . Now by the fundamental theorem of analysis

$$k(x) = \int_{l_x} \omega = \int_0^x h(t, y_0) dt$$

is a differentiable function in x and $k'(x_0) = h(x_0, y_0)$.

In a similar fashion one shows that f is differentiable in the y -direction and that $f_y(x_0, y_0) = g(x_0, y_0)$.

(ii) Consider the form $\omega(z) = g(z)dz$. Then ω is closed, and so by (i) there is a C^1 -smooth function f such that $df = \omega$. We have that

$$d\omega = f_z dz + f_{\bar{z}} d\bar{z}$$

Then we need to have $f_{\bar{z}} = 0$, so f is holomorphic, and $f_z = g$.

(Alternatively (but that is a different angle) we have that $g(z) = \sum_{j=0}^{\infty} c_j z^j$ is converging compactly on Δ . Then we have seen that $f(z) = \sum_{j=0}^{\infty} \frac{c_j}{j+1} z^{j+1}$ converges compactly on Δ and that $f' = g$).

For a domain $\Omega \subseteq \mathbb{C}$ we say that $H^1(\Omega, \mathbb{C}) = 0$ if the following holds: for any open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of Ω , and any collection of locally constant functions $f_{ij} : U_{ij} \rightarrow \mathbb{C}$ satisfying

$$f_{ij} + f_{jk} + f_{ki} = 0 \text{ on } U_{ijk},$$

for all $i, j, k \in I$, there are locally constant functions $f_i : U_i \rightarrow \mathbb{C}$ such that $f_i|_{U_{ij}} - f_j|_{U_{ij}} = f_{ij}$. (A function $f : U \rightarrow \mathbb{C}$, $U \subset \mathbb{C}$ open, is *locally constant* if for any point $z \in U$ there is an open neighborhood V of z such that f is constant on V . Equivalently, f is constant on each connected component of U .)

Problem 2.

- (i) Prove that if $\Omega, \Omega' \subseteq \mathbb{C}$ are domains, and there exists a homeomorphism $\phi : \Omega \rightarrow \Omega'$, then $H^1(\Omega, \mathbb{C}) = 0$ if and only if $H^1(\Omega', \mathbb{C}) = 0$.

(ii) Prove that $H^1(\Delta, \mathbb{C}) = 0$.

(Hint: Try to mimic the proof of Theorem 4 in Chapter 6 in Narasimhan's book.)

(iii) Prove that for any simply connected domain $\Omega \subseteq \mathbb{C}$ we have that $H^1(\Omega, \mathbb{C}) = 0$.

(iv) Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. Prove that if $\omega \in \mathcal{E}^1(\Omega)$ is closed, then there exists an $f \in \mathcal{E}(\Omega)$ such that $df = \omega$.

Solution:

(i) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of Ω and consider a collection of locally constant functions $f_{ij} : U_{ij} \rightarrow \mathbb{C}$ satisfying

$$f_{ij} + f_{jk} + f_{ki} = 0 \text{ on } U_{ijk},$$

for all $i, j, k \in I$. Assume that $H^1(\Omega', \mathbb{C}) = 0$. Now $\mathcal{U}' = \phi(\mathcal{U})$ is an open cover of Ω' and the functions $f'_{ij} = f_{ij} \circ \phi^{-1}$ satisfy

$$f'_{ij} + f'_{jk} + f'_{ki} = 0 \text{ on } U'_{ijk},$$

Consider functions f'_i as described above in the definition that $H^1(\Omega', \mathbb{C}) = 0$. Set $f_i = f'_i \circ \phi$.

There is a complete symmetry between Ω and Ω' , so this argument solves the problem.

(ii) Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of Δ and consider a collection of locally constant functions $f_{ij} : U_{ij} \rightarrow \mathbb{C}$ satisfying

$$f_{ij} + f_{jk} + f_{ki} = 0 \text{ on } U_{ijk}$$

As we have seen in class there are smooth functions $g_i : U_i \rightarrow \mathbb{C}$ such that $g_i|_{U_{ij}} - g_j|_{U_{ij}} = f_{ij}$. Then $dg_i = dg_j$ on U_{ij} and so the collection of dg_i 's defines a differential form ω on Δ which is closed. So there is a function f such that $df = \omega$. Set $f_i = g_i - f$. Then f_i is locally constant and $f_i - f_j = g_i - g_j$ on U_{ij} .

(iii) Any simply connected domain $\Omega \subset \mathbb{C}$ is homeomorphic to Δ ; if $\Omega = \mathbb{C}$ we may construct a homeomorphism by hand, and if not we may use the Riemann mapping theorem. Now it follows from (i) and (ii).

(iv) We may cover Ω by disks Δ_j , $j \in \mathbb{N}$. On each disk Δ_j there is a function g_j such that $dg_j = \omega|_{\Delta_j}$. But the collection $g_{ij} = g_i - g_j$ is locally constant (each defined on Δ_{ij}) and clearly

$$g_{ij} + g_{jk} + g_{ki} = 0 \text{ on } \Delta_{ijk}$$

By (iii) there is a family of locally constant functions h_i (each defined on Δ_i) such that $g_{ij} = h_i - h_j$ on Δ_{ij} for all i, j . Then the collection of functions $f_i = g_i - h_i$ defines a global function on Ω , and $df = \omega$.

Problem 3. In this problem we assume that $\Omega \subseteq \mathbb{C}$ is a domain with $H^1(\Omega, \mathbb{C}) = 0$.

- (i) Prove that for any $g \in \mathcal{O}(\Omega)$ there exists $f \in \mathcal{O}(\Omega)$ such that $f' = g$.
- (ii) Prove that for any $g \in \mathcal{O}^*(\Omega)$ there exists $f \in \mathcal{O}(\Omega)$ such that $e^f = g$.
- (iii) Prove that if $\Omega \neq \mathbb{C}$ then Ω is biholomorphic to Δ .

Solution:

- (i) Mimic the proof Problem 2 (iv) of using Problem 1 part (ii).
- (ii) There is a function $\tilde{f}$ such that $\tilde{f}' = \frac{g'}{g}$. Then

$$(e^{-\tilde{f}}g)' = -\tilde{f}'e^{-\tilde{f}}g + e^{-\tilde{f}}g' = 0$$

Then $e^{-\tilde{f}}g = c \neq 0$ and so $g = ce^{\tilde{f}} = e^{\log c + \tilde{f}} =: e^f$, where $\log$ is an arbitrary choice of logarithm of c .

(iii) By the Riemann mapping theorem (Second formulation) this follows if for any $g \in \mathcal{O}^*(\Omega)$ there exists $h \in \mathcal{O}(\Omega)$ such that $h^2 = g$. By (ii) there is f such that $e^f = g$. Set $h = e^{\frac{1}{2}f}$.