MAT4800

14 October 2026

§ 11. Linear Differential Equation

If $w: \mathbb{C} \to \mathbb{C}$ is a C^1-function, & $a \in \mathbb{C}$ a constant then the ODE:

$$\begin{cases} w' = aw \\ w(0) = w_0 \end{cases}$$

has a unique solution $w(z) = w_0 e^{az}$.

Now we want to extend this to higher dimensions:

$w: \mathbb{C}^n \to \mathbb{C}$

$A \in M_{n\times n}(\mathbb{C})$ an $n \times n$ linear matrix

& the ODE (Ordinary Differential Equation):

$$\begin{cases} w' = A.w \\ w(0) = w_0. \end{cases}$$

Example: $w = \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

& we consider the equation

$$w' = A \cdot w$$

$$\Leftrightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x+2y \\ 2x+y \end{pmatrix}$$

$$\Leftrightarrow \text{2 equations: } \begin{cases} x' = x+2y \\ y' = 2x+y \end{cases}$$

Actually, we will consider more general:
A needs not to be a constant matrix
but depending on z.

For example:

$$\begin{cases} x' = (1+z)x + (2+z^2)y \\ y' = (2-z)x + (1+z^3)y \end{cases}$$

Thm 11.2: Let $A: \mathbb{D} \to M_{n\times n}(\mathbb{C})$ be a holomorphic map, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < R\}$, $0 < R \leq \infty$

(Meaning $A = (a_{ij}(z))_{i,j=1,\dots,n}$, & $a_{ij}: \mathbb{D} \to \mathbb{C}$ are holomorphic functions.)

Then for each $w_0 \in \mathbb{C}^n$, $\exists$ one & only one $w: \mathbb{D} \to \mathbb{C}^n$ such that:

$$\begin{cases} w'(z) = A(z)\,w(z) & \forall z \in \mathbb{D} \\ w(0) = w_0 \end{cases}$$

Proof: Use the local existence & uniqueness of an ODE in the differentiable case. Then, use that $\mathbb{D}$ is simply connected & if $A(z)$ is holomorphic, then the solution to $\begin{cases} w'(z) = A(z)w(z) \\ w(0) = w_0 \end{cases}$ is also holomorphic.

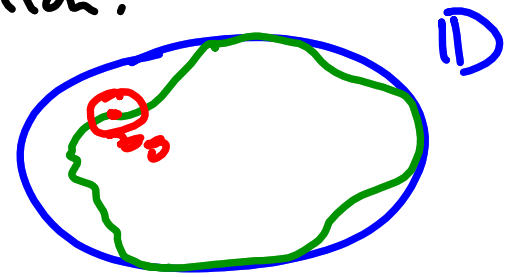
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More precisely: Picard - Lindelöf theorem. If we take $0 \in U \subseteq \mathbb{D}$ to be the biggest open set so that a solution exists then $U = \mathbb{D}$. Because if $U \neq \mathbb{D}$,

U is open $\Rightarrow$ let $z_0 \in \partial U \setminus \partial \mathbb{D} \neq \emptyset$

$\Rightarrow$ By Picard - lindelöf we can extend the solution to a neighborhood of z_0

$\Rightarrow$ we have bigger set than U a contradiction. □



Now this theorem can be extended to a simply connected Riemannian manifold (e.g. $\mathbb{P}^1$, $\mathbb{C}$, a disc in $\mathbb{C}$):

What we do is we look at open subsets U of X which are biholomorphic to $\mathbb{D}$, we solve the equation $w' = A(z)w(z)$ a U, & then we glue all these solutions together to get a global solution on X.

Setting: $w: X \to \mathbb{C}^n$

$A: X \to M_{n\times n}(\mathbb{C})$

holomorphic. Here we use simply-connectedness through Corollary 7.4:

Recall Corollary 7.4: RS

If X is simply connected, $a \in X$ & $\varphi \in \mathcal{O}_a$ is a function germ which accepts an analytic continuation along every curve starting at a. Then there is an $f \in \mathcal{O}(X)$ such that the germ of f at a is φ.

We apply this for $\varphi \equiv w$, & show that we can extend any solution of $w' = Aw$ along any bounded curve in X. But this is just the standard case of an ODE.

[Won't discuss about Facts of Automorphy ⇝ a little bit complicated.]

Bessel's equation:

$$w'' + \frac{1}{z} w' + \left(1 - \frac{\rho^2}{z^2}\right) w = 0$$

$\rho \in \mathbb{C}$ a constant.

Let's look at a solution of the form:

$$\varphi(z) = z^{\sigma} \sum_{n=0}^{\infty} c_n z^n, \quad c_0 \neq 0, \quad \sigma \in \mathbb{C}$$

(Actually a previous theorem shows that we have a solution of this form.)

So we will plug this in the equation to find σ & $c_0, c_1, \ldots$

$$w(z) = z^{\sigma} \sum_{n=0}^{\infty} c_n z^n$$

$$\Rightarrow w'(z) = \sigma z^{\sigma-1} \sum_{n=0}^{\infty} c_n z^n + z^{\sigma} \sum_{n=0}^{\infty} n c_n z^{n-1}$$

$$= z^{\sigma} \sum_{n=0}^{\infty} (\sigma + n) z^{n-1} c_n$$

similarly:

$$w''(z) = z^{\sigma} \sum_{n=0}^{\infty} (\sigma+n)(\sigma+n-1) c_n z^{n-2}.$$

Here be careful about how to take derivative of z^σ when σ is not an integer.
(But fortunately, everything is good).

Plug in the equation, we then get:

(i) $(\sigma^2 - p^2) c_0 = 0$

(ii) $((\sigma+1)^2 - p^2) c_1 = 0$

(iii) $((\sigma+n)^2 - p^2) c_n + c_{n-2} = 0$ $\forall\ n \geq 2$.

Now $c_0 \neq 0 \Rightarrow \sigma^2 - p^2 = 0 \Rightarrow \sigma = \pm p$.

These solutions are called Bessel's function. □