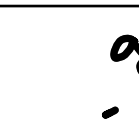
14 September 2026
MAT 4800

§4 (cont.)

Thm 4.9 (Holomorphicity: lifting of a holomorphic map is holomorphic map): Assume X, Y, Z are Riemann surfaces. $p: Y \to X$ an unramified (i.e no critical point, non-constant) holomorphic map & $f: Z \to X$ holomorphic. If f has a lifting $g: Z \to Y$, then g is also holomorphic.

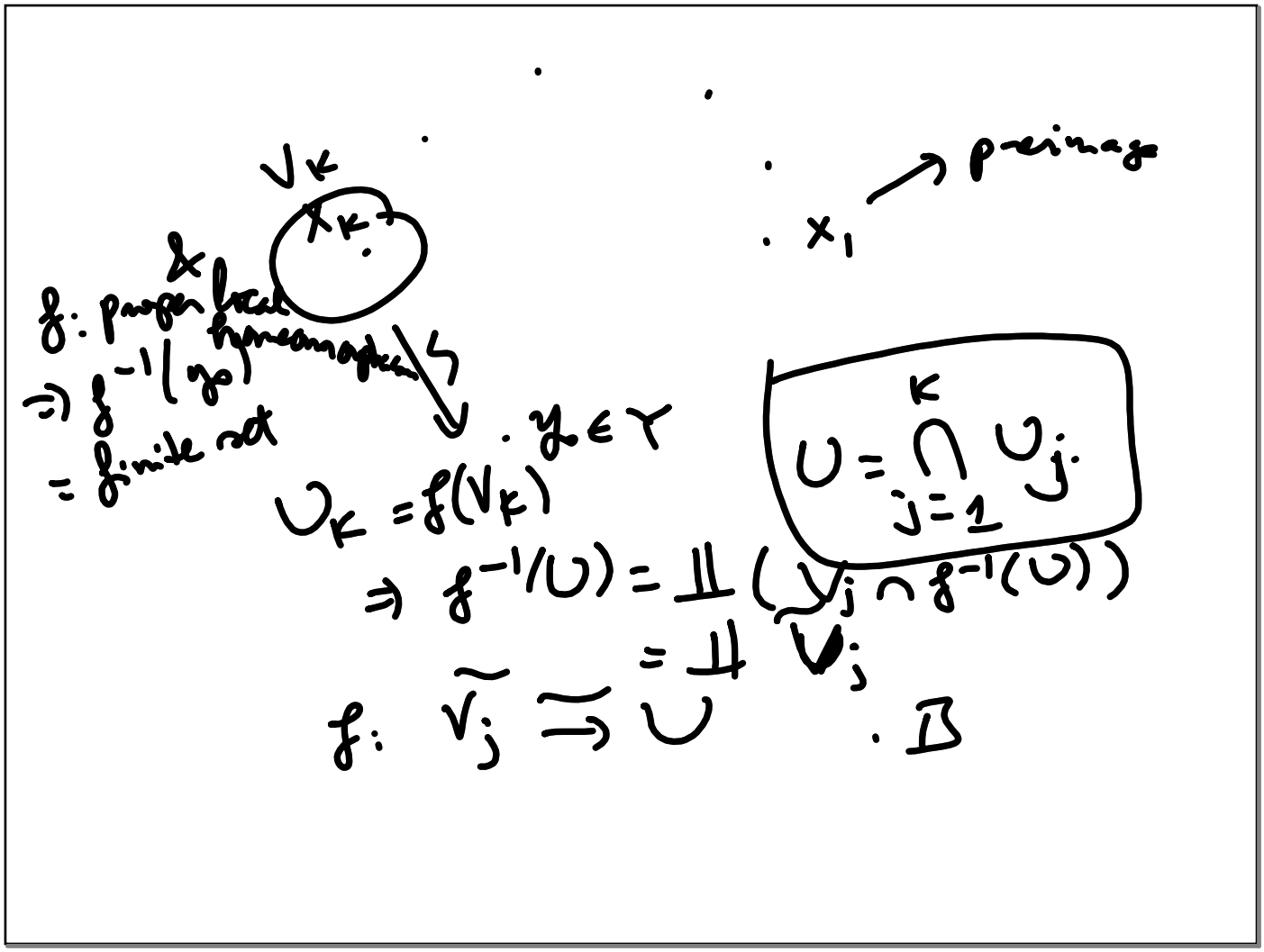


Proof: Locally, an unramified holomorphic map between RS (they have the same dim = 1) is biholomorphic. So locally: g is biholomorphic to f, & hence is holomorphic. Since holomorphic is a local property ⇒ g is holomorphic (globally). □

Remark: In general a lifting may not exist. A case when it is guaranteed to exist is when Z = an interval (or simply connected), plus a condition (covering map). □

Now use the Lecture notes on 26 September 2023. Use covering map def below.

Def 4.11 (Covering map) $f: X \to Y$ is a covering map if $\forall\, y_0 \in Y$, $\exists\, y_0 \in U \subseteq Y$ so that $f^{-1}(U) = \coprod_{j \in J} V_j$, → maybe infinite & $f|_{V_j}: V_j \to U$ is homeomorphic. □


V_K
x_K
x_1
preimage
f: proper local homeomorphism
$\Rightarrow f^{-1}(y_0)$
= finite set
$y \in Y$
$U_K = f(V_K)$
$U = \bigcap_{j=1}^{K} U_j$
$\Rightarrow f^{-1}(U) = \coprod (V_j \cap f^{-1}(U))$
$= \coprod V_j$
$f: V_j \xrightarrow{\sim} U$
$\square$

Thm 4.10 (lifting of homotopy of curves):

X, Y : Hausdorff.

$p: Y \to X$ a local homeomorphism.

$I = [0,1]$

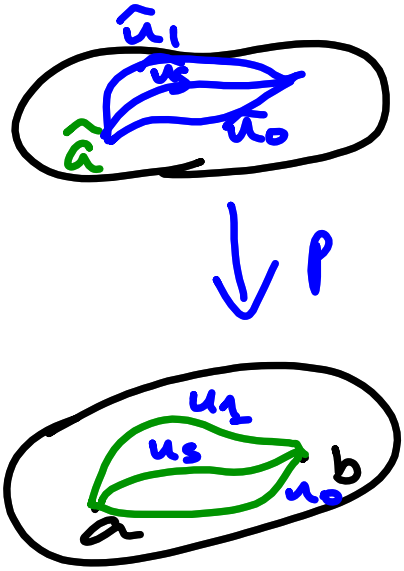
$A: I \times I \to X$ is such that

$A(0,s) = a$ & $A(1,s) = b$ $\forall s \in I$.

Set $u_s(t) := A(t,s)$

If every curve u_s can be lifted to a curve $\widehat{u_s}$ with $\widehat{u_s}(0) = \widehat{a}$ $\forall s$, then $\widehat{u_0}$ & $\widehat{u_1}$ satisfy: $\widehat{u_0}(1) = \widehat{u_1}(1)$ & they are homotopic.

Proof:



First step: Show that $\forall s \in I$ then:

$$\tilde{u}_s(1) = \tilde{u}_0(1).$$

Claim 1: $\exists\, \varepsilon > 0$ so that

$$\tilde{A} : I \times I \to Y,$$

$$\tilde{A}(t, s) = \tilde{u}_s(t)$$

is continuous for $(s,t) \in [0,\Sigma) \times I$.

Proof of Claim 1: Use that p is a local homeomorphism & the lifting is unique (if exists). First continuous on $[0,\Sigma] \times [0,\delta_1]$, then use local existence to extend to $\delta_2 > \delta_1$.

Claim 2: A is continuous on $I \times I$.

Proof: Similar to above. First it is continuous on $[0,\Sigma) \times I$. Then use local lifting & uniqueness to extend to $\Sigma_1 > \Sigma$ ... ∎

Thm 4.17 (generalisation of Theorem 4.14)

$$\begin{array}{ccc} & & F \\ & \exists \nearrow & \downarrow p \Rightarrow \text{covering} \\ \text{Simply connected} \to Z & \xrightarrow{f} & X \end{array}$$

Step 3: Show that

$\widehat{A}(1,s) = \hat{b}$ for all s,

for $\hat{b} \in p^{-1}(b)$.

Proof: $\widehat{A}(1,\overline{I})$ is connected (because I is connected & $\widehat{A}$ is continuous).

Also $\widehat{A}(1,I) \subseteq p^{-1}(b)$

$\Rightarrow$ $\widehat{A}(1,I)$ must be 1 point, which we call $\hat{b}$. □

In particular, combining Theorem 4.14 (in the 2023 lecture notes) & Thm 4.10 we obtain

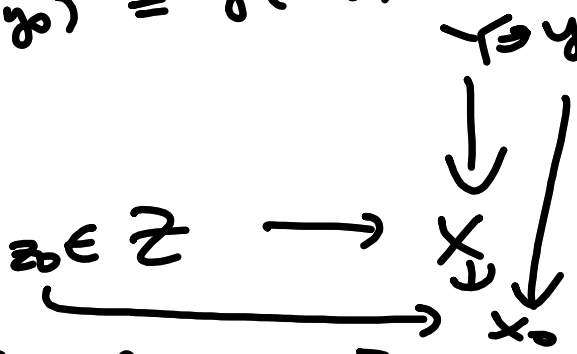
Cor: If $p: Y \to X$ is covering map, & c_1 & c_2 are 2 homotopic curves on X, then liftings of c_1 & c_2 to Y exist if we choose initial point in p^{-1}(of initial point of c_1) & the lifting curves are homotopic. □

Universal covering contains every covering.

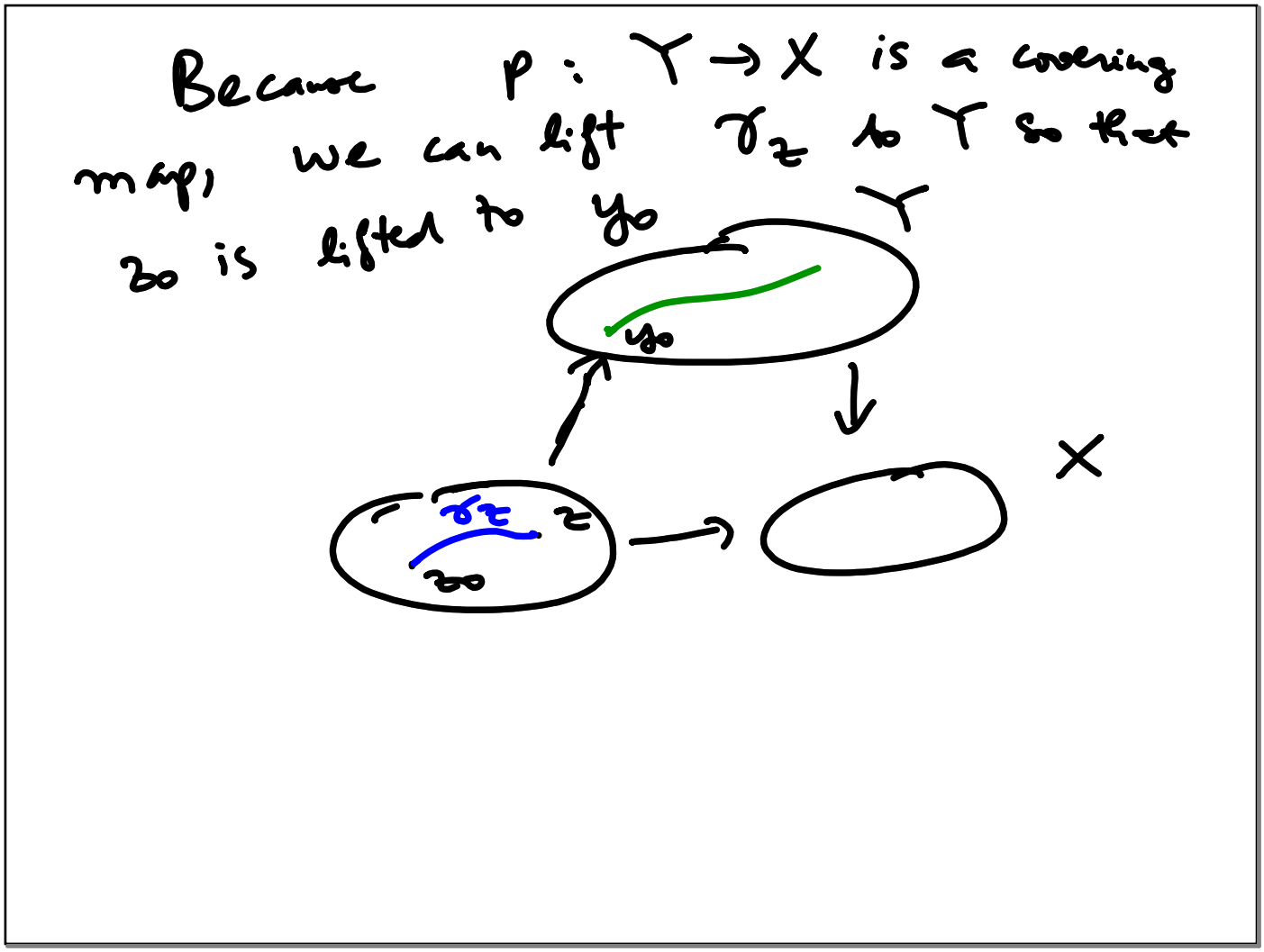
Proof: Idea:

Fix a point $z_0 \in Z$.

Choose a point $y_0 \in Y$ so that $p(y_0) = f(z_0)$.

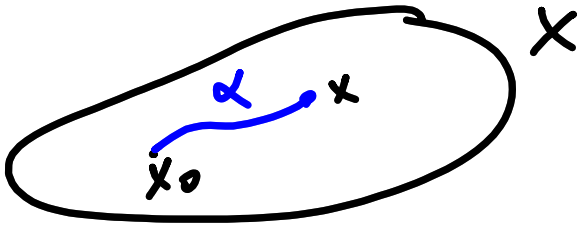


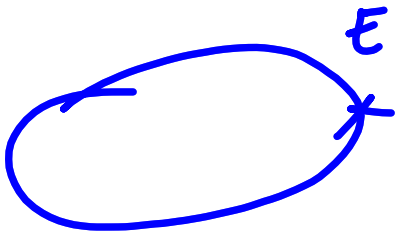
$\forall\, z \in Z$, because Z is path-connected, $\exists$ a curve $\gamma_z \subseteq Z$ which connects z_0 to z.


Because $p: Y \to X$ is a covering map, we can lift γ_z to Y so that z_0 is lifted to y_0
Y
y_0
X
γ_z
z
z_0

By Thm 4.10 (Lifting of homotopy of curves) if we have another curve γ_z', then the end points of the lifting of γ_z & γ_z' are the same.

So this gives a well-defined lifting of f to Y. We only need to prove that it is continuous. But this comes from local lifting of p & by local homeomorphism that Z is connected. □





Example at the end of lecture note 26 September 2023:

$$E = \{ [X:Y:Z] \in \mathbb{P}^2 : Y^2Z = X^3 + Z^3 \}$$

$E_1 = (\text{affine part of } E) = E \cap U_1$,

$U_1 = \{ [X:Y:Z] : Z \neq 0 \}$

$E_2 = \{[X:Y:Z]:$ ~~$[X:Y:Z]$~~$\}$.

$= E \cap U_2$, $U_2 = \{[X:Y:Z] : Y \neq 0\}$.

On U_1, choose coordinates:

$$x_1 = \frac{X}{Z}, \quad y_1 = \frac{Y}{Z}$$

On U_2, choose coordinates

$$x_2 = \frac{X}{Y}, \quad z_2 = \frac{Z}{Y}.$$

Now the map $\pi_1 : E_1 \to \mathbb{C}$

$(x_1, y_1) \mapsto$ ~~ϕ~~ x_1

holomorphic.

What is this π_1 on E_2?

$$\pi_2 : E_2 \xrightarrow{??} \mathbb{C} \quad (??)$$

$$(x_2, z_2) \mapsto ??$$

$$x_1 = \frac{X}{Z}, \quad y_1 = \frac{Y}{Z}$$

$$x_2 = \frac{X}{Y}, \quad z_2 = \frac{Z}{Y}$$

$$\Rightarrow x_1 = \frac{X}{Z} = \frac{X/Y}{Z/Y} = \frac{x_2}{z_2}$$

The map π_2 is (provided $z_2 \neq 0$) then it is

$$(x_2, z_2) \mapsto \frac{x_2}{z_2}.$$

So it has a singularity at $z_2 = 0$.

So $E_2 = \{ 1 = x_2^3 + z_2^3 \}$

$x_2^3 = -1$ (3 points).

HW: Show that the singularity is poles.

By theorem we learned before, π_2 can be extended holomorphically to $\mathbb{P}^1$:

$$\pi_2 : E_2 \to \mathbb{P}^1.$$

HW: $E_3 = E \cap U_3$,

$$U_3 = \{[X:Y:Z] : X \neq 0\}$$

Show that π_1 can be extended to a holomorphic map:

$$\pi_3 : E_3 \to \mathbb{P}^1.$$

Together, π_1, π_2, π_3 gives us a holomorphic, non-constant $\pi : E \to \mathbb{P}^1$.

HW: Show that π is proper, compute the degree of π.