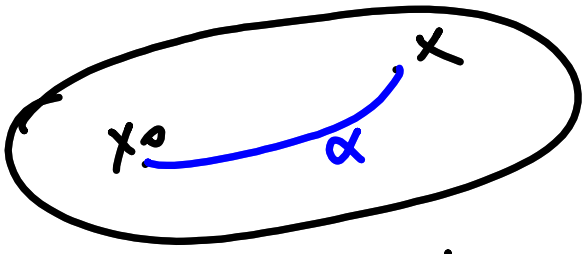
MAT 4800
16 September 2026

Fix $x_0 \in X$
(base point



* If $\alpha \sim \alpha'$ $\Rightarrow$ $U[\alpha] \cap U[\alpha'] = \emptyset$

homotopic

$\mathcal{U}$ = simply connected

$x \xrightarrow{R} z$ is homotopic to $\alpha * \alpha'$

Assume $U[\alpha] \cap U[\alpha'] \neq \emptyset$, show that $\alpha \sim \alpha'$



$\Rightarrow \quad x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\gamma} y$

& $x_0 \xrightarrow{\alpha'} x_2 \xrightarrow{\gamma'} y$

are homotopic.

Since $\mathcal{U}$ is simply connected, γ^*

$\exists$ "$x^* \in \mathcal{U}$ & a curve $x_2 = x^* \to y$

in $\mathcal{U}$ so that both $x_1 \xrightarrow{\gamma} y$ & $x_2 \xrightarrow{\gamma'} y$

are homotopic to $x^* \xrightarrow{\gamma^*} y$.

So $x_0 \xrightarrow{\alpha} x_1 \xrightarrow{\gamma} y \sim x_0 \xrightarrow{\alpha} x_1 \to x_2 \xrightarrow{\gamma^{-1}} y$

$x_0 \xrightarrow{\alpha} x_2 \xrightarrow{\gamma} y \sim x_0 \xrightarrow{\alpha'} x_2 \xrightarrow{\gamma} y$

So $x_0 \xrightarrow{\alpha} x_1 \to x_2 \sim x_0 \xrightarrow{\alpha'} x_2$

& $x_1 \to x_2$ homotopic to a point $\Rightarrow \alpha \sim \alpha'$. ∎

If $\mathcal{U}$ simply connected in X:

$$p^{-1}(\mathcal{U}) = \bigcup_{\substack{x \in \mathcal{U} \\ \alpha}} \mathcal{U}[x,\alpha]$$

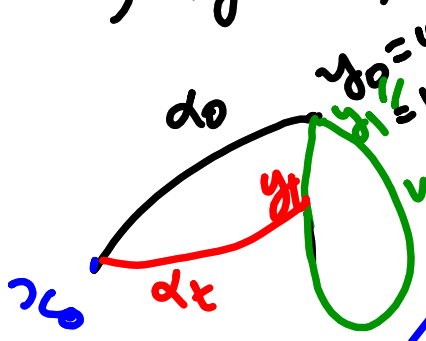
$$= \bigcup_{\substack{x^* \in X \\ \alpha}} \mathcal{U}[x^*,\alpha] \,.$$

x^* any point in $\mathcal{U}$.

* $\tilde{X}$ simply connected:

$(\beta_1, \alpha_1) \in X$ & a curve φ : [redacted] with the end point (β_1, α_1)

$\varphi = \{(\gamma_t, \alpha_t) : \gamma_{t,1} = \gamma_{t,0},\ \alpha_1 = \alpha_0\}$



$\varphi =$ a closed curve on X $\in \pi_1(X, x_0)$

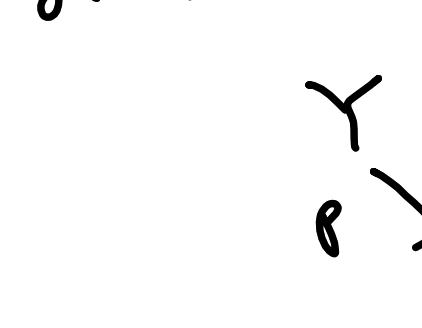
+ a homotopy ... with end points (x_0, y_t)

call it u in the proof

* Deck transformation:

$p : Y \to X$ is a covering map.

$f: Y \to Y$ is in $\mathrm{Deck}(Y/X)$

if 

f : homeomorphic

$p \circ f = p$

(i.e: f preserves fibers of p).

If $y \in p^{-1}(x) \Rightarrow p(f(y)) = p(y) = x$

$\Leftrightarrow f(p^{-1}(x)) \subseteq p^{-1}(x)$.

Actually $f(p^{-1}(x)) = p^{-1}(x)$

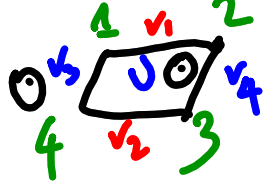
because f is a homeomorphism. □

Example for $\pi_1(X) = \mathrm{Deck}(\tilde{X}/X)$: .

$X = \mathbb{C}/\Lambda$ elliptic curve

Then projection $p: \mathbb{C} \to \mathbb{C}/\Lambda$ is covering map

$p^{-1}(U)$ So $\pi_1(X) = \text{Deck}(\mathbb{C}/E)$



Now what is $\text{Deck}(\mathbb{C}/E)$?

($\Rightarrow$ Since $\mathbb{C}$ & E are complex manifold, a $g \in \text{Deck}(\mathbb{C}/E)$ is an automorphism of $\mathbb{C}$ & $g(\Lambda) \subseteq \Lambda$.

What is an automorphism of $\mathbb{C}$?

If $f: \mathbb{C} \to \mathbb{C}$ is an automorphism

$\Rightarrow$ f is holomorphic / entire

~~So f must be~~

What would be the singularity of f at ∞?

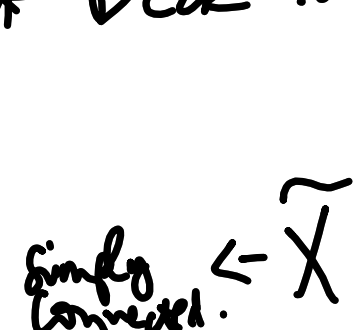
First, it cannot be essential singularity. (HW). Also cannot be removable. $\Rightarrow$

Must be a pole of order 1. $\Rightarrow f(z) = az + b$, $a \neq 0$, $a, b \in \mathbb{C}$

Now need condition $f(\Lambda) \subseteq \Lambda$

HW: If $\Lambda = \mathbb{Z} \oplus w\mathbb{Z}$, where $w \neq 1$, $w^3 = 1$, what is $\mathrm{Deck}(\mathbb{C}/E)$?

* Deck transformation as lifting:

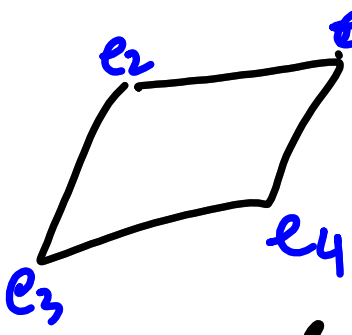


simply connected.

To get $\tilde{F}$ we need: 2 points $x_1 \in \tilde{X}$, $x_2 \in \tilde{X}$ so that $p(x_1) = p(x_2)$ & $F(x_1) = x_2$. (See §4).

Example: Look back to Deck(~~X/X~~ C/E)

For example, if we want to get $F \in \mathrm{Deck}(\mathbb{C}/\Gamma)$



Γ

$\Rightarrow$ Need to choose $e \in \Lambda$ & define $F(e_1) = e$

$\Lambda = \mathbb{Z} \oplus i\mathbb{Z}$: Example:

$e_1 = 0$, $e = 1+i$ $\quad F(0) = 1+i$

$F(z) = az + b$, $a \neq 0$

$F(0) = 1+i \Rightarrow b = 1+i$

$F(\Lambda) \subseteq \Lambda \Rightarrow a(\mathbb{Z} \oplus i\mathbb{Z}) \subseteq \mathbb{Z} \oplus i\mathbb{Z}$

(a) $u+iv,\ u,v \in \mathbb{R}$

$(u+iv)(x+iy) = x'+iy',$

$\forall x,y \in \mathbb{Z} \Rightarrow x', y' \in \mathbb{Z}$

$x=1, y=0$

$\Rightarrow u+iv \in \mathbb{Z}+i\mathbb{Z}$

$x=0, y=1$

$\Rightarrow -v+iu \in \mathbb{Z}+i\mathbb{Z}$

$u, v \in \mathbb{Z}$

F^{-1} also in $Deck(\mathbb{C}/F)$

$F(z) = (u+iv)z + 1+i$

$\Rightarrow F^{-1}(z) = \dfrac{z-(1+i)}{u+iv}$

$$F^{-1}(z) = \frac{z}{u+iv} - \frac{1+i}{u+iv}$$

$$\Rightarrow \begin{cases} \frac{1}{u+iv} \in \mathbb{Z}+i\mathbb{Z} \\ u+iv \in \mathbb{Z}+i\mathbb{Z} \end{cases}$$

$$\Rightarrow (u,v) = \pm(1,0),\ (0,1)$$