MAT 4800
21 October 2026

dx.

Čech cohomology is:

* isomorphic to the usual ones (e.g. singular cohomology, de Rham cohomology, ...)
* easy to compute (e.g. try to compute $H^1(S^1, \mathbb{R})$ using de Rham cohomology w.s. Čech cohomology. S^1 ↓ circle)
* Easy to operate (e.g. pullback, push forward, with compact support = use in Leray spectral sequence)

* Allow to establish theoretical results (e.g. Leray spectral sequence)

* Allow to extend to cases where continuous method cannot be applied.

* In the category of open sets, then ~~$U \to V$~~ $V < U$ if V is smaller open set than U.

In the category of open coverings:
$\mathcal{V} < \mathcal{U}$ if open sets in $\mathcal{U}$ are bigger than open sets in $\mathcal{V}$.

* $\underline{\mathbb{R}_X}$ presheaf

$\mathbb{R}_X(U) = \mathbb{R} \quad \forall\, U \subseteq X$ open $\neq \emptyset$

$\underline{\mathbb{R}_X}$: sheafify

$\underline{\mathbb{R}}_X(U) = \mathbb{R}^{\oplus \#\text{ of connected components of } U}$

$H^j(X, \mathbb{R}) \simeq H^j(X, \underline{\mathbb{R}_X})$

↑ de Rham, singular cohomology

↓ constant sheaf with stalk $\mathbb{R}$.

If $\phi \neq V \subseteq U$ open

then $\underline{\mathbb{R}}_X(U) \to \underline{\mathbb{R}}_X(V)$

$\| \qquad\qquad \|$

$\mathbb{R} \qquad\qquad \mathbb{R}$

identity map!!

$\underline{\mathbb{R}}_X(\phi) = \{0\}$.

* Example of morphism between sheaves

$\underline{\mathbb{R}}_X$ = constant sheaf

$\underline{\Omega}^j_X$ = the sheaf associated to the presheaf

Ω_X^j (presheaf) $U \subseteq X$ open

$\rightarrow \Omega_X^j(U) = \{ \omega : a\ C^\infty \text{ differentiable}$
$j\text{-}$ form over U,
$\omega = \sum f_{i_1,\dots i_j}\, dx_{i_1} \wedge \dots \wedge dx_{i_j} \}$

$\underline{\mathcal{C}}_X$ = sheaf associated to presheaf
$C_X(U) = \{ f : U \to \mathbb{R},\ C^\infty$ differentiable$\}$

Then $\forall$ U open there is a differentiation:

$$d: \Omega_X^j(U) \to \Omega_X^{j+1}(U)$$

$$(\ C_X = \Omega_X^0(U)\ )$$

$\Rightarrow$ it gives a morphism:

$$d: \underline{\Omega}_X^j \longrightarrow \underline{\Omega}_X^{j+1}$$

(used in de Rham cohomology).

* Category theory: Yoneda's lemma

..

* Remark: When working with maphisms of sheaves: reduce proofs to stalks.

* Exponential exact sequence:

$(X = RS)$

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

$$\alpha \longmapsto 2\pi i \alpha$$

$$\beta \longmapsto \exp(\beta)$$

Suffices to prove with a small open sets $U \subseteq X$, in particular: $U \simeq \mathbb{D} \subseteq \mathbb{C}$.

$$0 \to \mathbb{Z}_X(U) \to \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$$

$\mathbb{Z}$; $\{f : U \to \mathbb{C} \to \mathbb{C}^*\}$ holomorphic

$$0 \to \mathbb{Z} \to \mathcal{O}_X(U) \to \mathcal{O}_X^*(U) \to 0$$

$$\mathcal{O}_X(U) = \{ f : U \to \mathbb{C} \mid \text{holomorphic} \}$$

$$\mathcal{O}_X^*(U) = \{ g : U \to \mathbb{C} \mid \text{hol, } g \text{ never } 0 \}$$

$$\alpha \mapsto 2\pi i \alpha \quad (\text{constant function on } U)$$

$$f \mapsto \exp(f)$$

$$\ker\left(\mathcal{O}_X(U) \to \mathcal{O}_X^*(U) \right)$$

$$= \{ f : U \to \mathbb{C} \mid \text{hol}, \ \exp(f) = 1 \}$$

(1 ← identity element of $\mathcal{O}_X^*(U)$)

f takes discrete values $2\pi i \mathbb{Z}$, & f cont.

$\Rightarrow f = \text{const } 2\pi i k$, some $k \in \mathbb{Z}$.

$\mathcal{O}_X(U) \to \mathcal{O}_X^*(U) \to 0$ is exact because of the existence of logarithm theorem.

* Conclusion:
  Either work with stalks
  or work with very good open subsets (like simply connected ones)

* In the analytic setting:

Simply-connected open subsets are what we should use.

In the algebraic setting: affine open subsets are what we should use.

$U \subseteq \mathbb{A}^N$ closed subset

* Exercise: Show that

$(\mathcal{O}_X^*)_p = \{ f$ a Taylor series on a non-empty open subset at p, with: $\sum_{n \geq 0} a_n z^n$ & $a_0 \neq 0 \}$

* Partition of unity (technique in analysis):

Let $\{U_i\}_{i \in I} = \mathcal{U}$ be an open covering.

$\Rightarrow \exists \quad f_i : U_i \to [0, 1]$

such that $\operatorname{supp}(f_i) \subseteq U_i$

& $\sum_{i \in I} f_i = 1$ (partition of unity. □)

We use this to write any object g & $g = \sum_{i \in I} g_i$, where $\operatorname{supp} g_i$ $\operatorname{supp}(g_i) \subseteq U_i$.

It allows us to glue together local subjects to get a global subject.

* When proving some results for $\mathbb{C}_X$ $\Rightarrow$ use the $\mathbb{C}_X \hookrightarrow \mathcal{E}$

When proving some results for $\mathbb{Z}_X$, then use the SES:

$$0 \to \mathbb{Z}_X \to \mathbb{C}_X \to \mathbb{C}_X^* \to 0$$

* For simply connected set:
Use Poincaré lemma.