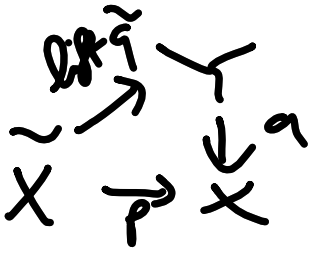
21 September 2026

MAT 4800

2 applications of universal covering

$p: \tilde{X} \to X$

(1) Any covering map $q: Y \to X$ "comes" from the universal covering



(2) $\pi_1(X, x_0)$ can be calculated from the covering / deck transformation.

* $z \in \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$

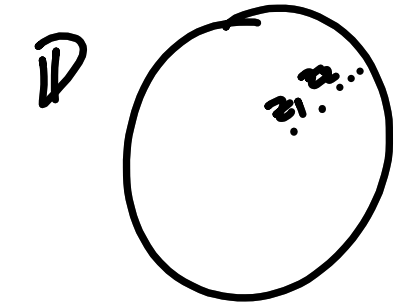
$w_0 \in \mathbb{D}$, $w_0 \neq 0$, want to solve

$$e^{-\left(\frac{1+z}{1-z}\right)} = w_0 \quad (1)$$

Write $w_0 = r e^{i\theta}$, $0 < r < 1$

$$w_0 = e^{\ln r + i\theta}$$

$$\Rightarrow (1) \Leftrightarrow -\left(\frac{1+z}{1-z}\right) = \ln r + i\theta + 2n\pi i \text{ for } n \in \mathbb{Z}.$$



The solutions to

$$-\left(\frac{1+z}{1-z}\right) = \ln n + i\theta + 2n\pi i, \quad n \in \mathbb{Z}, \quad z \in \mathbb{D}$$

is a sequence converging to $\partial\mathbb{D} = \{z : |z| = 1\}$

$\Rightarrow p$ is proper.

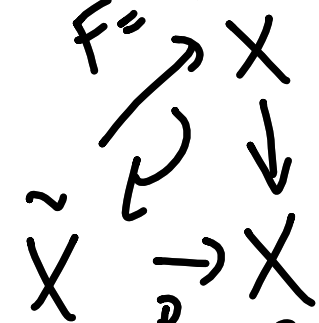
So to show that this map is a covering map, we can do as follows.

For each j, choose $z_j \in U_j \subseteq \mathbb{D}$, U_j does not contain any other z_k, $k \neq j$.

Then try to show that $p|_{U_j} : U_j \to \mathbb{D}^*$

is injective!!

* Deck transformation lift of p



* $E = \{ Y^2Z = X^3 + Z^3 \} \subseteq \mathbb{P}^2$ elliptic curve

$\pi : E \to \mathbb{P}^1$, $[X:Y:Z] \mapsto [X:Z]$.

Note: $[0:1:0] \in E$, but $\pi([0:1:0]) = [0:0] \notin \mathbb{P}^1$?

Actually $\pi([0:1:0])$ is well-defined point in $\mathbb{P}^1$

However, formally, it is not defined at [0:1:0], so we need to use classification of singularity & Limit Calculation to find the correct image!!

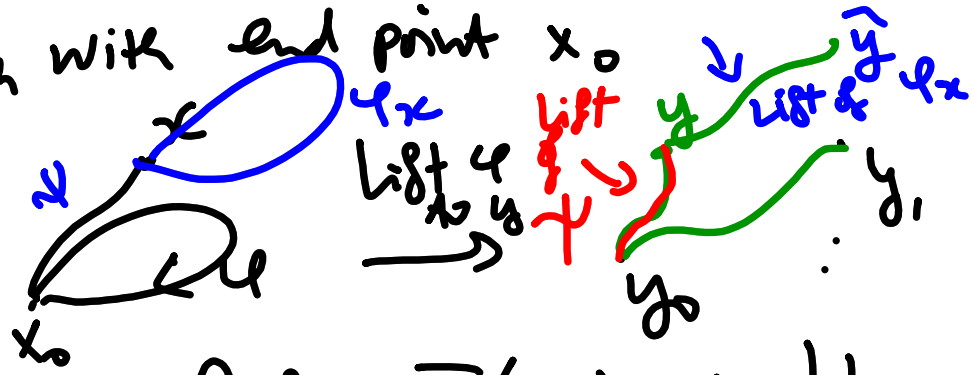
Choose coordinate chart $x_2 = X/Y$, $z_2 = Z/Y$

$\Rightarrow$ the curve becomes $z_2 = x_2^3 + z_2^3$,

& we want $\lim_{(x_2,z_2) \in \text{the curve},\ (x_2,z_2) \to (0,0)} \frac{x_2}{z_2}$.

First, claim that near $(0,0)$ then x_2 is a coordinate for the curve (1-dim complex)

* φ a closed curve in X based with end point x_0



Define $\tilde{F}(y_0) = y_1$!!

$F(y) = \tilde{y}$.

Continuity of F is easy.

To show that F has an inverse, let

$\tilde{F}$ = the lift of φ^{-1}.

* Elliptic curve:

$$E = \mathbb{C}/\Lambda$$

then $\mathrm{Deck}(\mathbb{C}/E)$

$$\simeq \Lambda \simeq \mathbb{Z} \oplus \mathbb{Z}$$

$$\rightarrow \pi_1(E) = \Lambda$$

$f \in \mathrm{Deck}(\mathbb{C}/E)$

$$\Rightarrow f(z) - z \in \Lambda \quad \forall z \in \mathbb{C}$$

f: holomorphic, Λ: discrete

$$\rightarrow f(z) - z \equiv \text{a constant} \in \Lambda.$$

$\Rightarrow f(z) = z + a, \quad a \in \Lambda.$

Vice versa if $f(z) = z + a$,

$a \in \Lambda \Rightarrow f \in \mathrm{Deck}(\mathbb{C}/E)$. □

* $f(z) = e^z$

$: \mathbb{C} \to \mathbb{C}^*$ is covering

Hint: Check it is proper.

Check it is local homeomorphism

$(\Leftrightarrow) \quad f'(z) \neq 0 \quad \forall z.$

$\Rightarrow$ Apply Theorem 4.22. □

* Let $A \subseteq \mathbb{C}$ be a finite set. Then any (holomorphic) automorphism f of $\mathbb{C} \setminus A$ is extended to a holomorphic automorphism of $\mathbb{P}^1$. (Basically, show that f has no essential singularity. ⟶ Use Casorati-Weierstrass theorem.

Why presheaf? Later we have a sheaf on X, but then we want to transform it to another space Y, & we not get a sheaf but only presheaf.

§6.5. Stalks of a presheaf

(We will apply for example to germ of holomorphic functions)

Roughly speaking, stalks of a presheaf are the values of the presheaf at points in X.

Recall: A presheaf is only defined on open subsets of X. A point $x_0 \in X$ is generally not open set, so it is not obvious how we can define the

value of a presheaf at x_0.

A rough idea: If $\exists$ U_0 open $\subseteq X$, $x_0 \in U_0$ & for all $U \subseteq X$ open & $x_0 \in U \Rightarrow U_0 \subseteq U$, we can define:

$$\mathcal{F}_{x_0} := \mathcal{F}(U_0).$$

But this case will almost never happen in interesting applications, so we need to define a kind of "smallest open set containing x_0", by using inverse limits!

Here is the precise idea:

Look at $\coprod_{X \supseteq U \text{ open}} F(U)$
$U \ni x_0$

Now define an equivalent relation on this as follows: $f_1 \in F(U_1)$ is equivalent to $f_2 \in F(U_2)$ iff $\exists$ open $W \subseteq U_1 \cap U_2$ with $x_0 \in W$ & $f_1|_W = f_2|_W$.

Another way to say: $f_1 \sim f_2$
$\iff$ f_1 & f_2 restricts to the same element near x_0.

Then we define the stalk at x_0:

$$\mathcal{F}_{x_0} = \varinjlim_{\substack{U \text{ open} \\ U \ni x_0}} \mathcal{F}(U) \Big/ \sim .$$

Remark: If $U \ni x_0$, then $\exists$ a map $\rho_{U,x_0}: \mathcal{F}(U) \to \mathcal{F}_{x_0}$

Example (HW): $X = RS$

$U \subseteq X \Rightarrow \mathcal{O}_X(U) =$ the set of holomorphic functions $f: U \to \mathbb{C}$.

- Show that it is a presheaf.
- What is the stalk at a point x_0?

(Hint: ~~it is the set~~ Near x_0, choose a local chart so that $X \simeq \mathbb{D}(0,\varepsilon) \subseteq \mathbb{C}$, then (here x_0 is 0)

$\mathcal{O}_{X,x_0} \simeq$ power series with positive radius of convergence

($F = \mathcal{O}_X$)