MAT 4800
23 September 2026

Main point: Any sheaf is a presheaf.
Any presheaf gives rise to a unique sheaf.

Def (Sheaf) A presheaf $\mathcal{F}$ on X is a sheaf if it satisfies Sheaf Axioms:

If $\{U_i\}_{i\in I}$ are open sets in X & $U = \bigcup_{i\in I} U_i$:

Axiom 1 (Uniqueness): If $f, g \in \mathcal{F}(U)$ & $f|_{U_i} = g|_{U_i}\ \forall i \Rightarrow f = g$. (Local equality implies global equality)

Axiom 2 (Existence): If $\forall i$, $\exists\, f_i \in \mathcal{F}(U_i)$ such that $\forall i,j$:

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j},$$

(only local sections)

then $\exists\, f \in \mathcal{F}(U)$ such that $\forall i \in I$:

$$f|_{U_i} = f_i.$$

(global section)

(i.e. local existence & compatibility $\Rightarrow$ global existence).

Remark: f in Axiom 2 is unique because of Axiom 1.

$\mathcal{F}(\phi)$ has a unique element.

(Why: if $U = \emptyset$, then .)

Example: $X = \{a, b\}$ discrete topology.

$\mathcal{F}$ presheaf on X with

$\mathcal{F}(\emptyset) = 0$, $\mathcal{F}(U) = \mathbb{Z}$ for $\emptyset \neq U \subseteq X$. Restriction is identity if $V \neq \emptyset$, & the map if $V = \emptyset$.

Is $\mathcal{F}$ a sheaf?

$$0 = \mathcal{F}(\emptyset) \leftarrow \mathcal{F}\{a\} = \mathbb{Z} \leftarrow \mathcal{F}\{a,b\} = \mathbb{Z}$$
$$\mathcal{F}\{b\} = \mathbb{Z}$$

(id, id)

It is not a sheaf because

if we choose $U_1 = \{a\}$, $U_2 = \{b\}$

$f_1 = 1 \in F(U_1) = \mathbb{Z}$

$f_2 = 2 \in F(U_2) = \mathbb{Z}$

then there is no $f \in F(U_1 \cup U_2) = F(X) = \mathbb{Z}$

so that $f|_{U_1} = f_1$ & $f|_{U_2} = f_2$.

Question: How can we get a sheaf from a presheaf?

The topological space associated to a presheaf (also called étale space):

F presheaf over X

$|F|$ = étale space of F

$$= \coprod_{x \in X} F_x .$$

So an element y of $|F|$ consists of a point $x \in X$ & an element f_x of F_x. a $y = (x, f_x)$.

germ of F at x.

Then we can define

$$\rho : |\mathcal{F}| \to X$$

$$\rho = (x, f_x) \longmapsto x.$$

We Define The topology on $|\mathcal{F}|$ to have a basis

$$[U, f] = \{\rho = (x, \rho_x(f)) : x \in U\}$$

where $U \subseteq X$ is open
& $f \in \mathcal{F}(U)$ is a section of $\mathcal{F}$ over U.

To show that it is a topology need to check that $[U_1, f_1] \cap [U_2, f_2]$ is the union of $[U_j, f_j]_{j \in J}$.

It suffices to show that if

$\varphi_{x_0} = (x_0, f_{x_0}) \in [U_1, g_1] \cap [U_2, g_2]$

$\Rightarrow \exists$ U an open subset of X containing x_0 & $g \in \mathcal{F}(U)$ so that:

$\rho_{x_0}(g) = f_{x_0}$.

This is just from the definition !!

If $\varphi_0 = (x_0, f_{x_0}) \in [U_1, g_1]$

$\Leftrightarrow \rho_{x_0}(g_1) = f_{x_0}$. Similarly

$\rho_{x_0}(g_2) = f_{x_0}$. By definition of 'stalk' $\Rightarrow \exists$ $V \subseteq U_1 \cap U_2$

such that $f_1|_V = f_2|_V$. Call $g = f_1|_V$.

But then: $f_{x_0} \in [V_1, g] \cap [V_2, g]$.

Def (Identity theorem): A presheaf $\mathcal{F}$ on X satisfies the identity theorem if $\forall$ connected open set $T \subseteq X$, $\forall f, g \in \mathcal{F}(T)$, $\exists a \in T$ such that $f_a = g_a$ (stalk) $\Rightarrow f = g$.

Thm (6.2.10): Assume X locally connected Hausdorff space, $\mathcal{F}$ presheaf on X satisfying the Identity theorem. Then $|\mathcal{F}|$ – étale space – is Hausdorff.

Proof: Let $\varphi_1 = (x_1, f(x_1))$ & $\varphi_2 = (x_2, f(x_2))$ different elements of $|\mathcal{I}|$. Need to show that there are $[U_1, g_1]$ & $[U_2, g_2] \leq \mathcal{I}$ with:

$$\begin{cases} [U_1, g_1] \cap [U_2, g_2] = \emptyset \\ \varphi_1 \in [U_1, g_1] \\ \varphi_2 \in [U_2, g_2] \end{cases}$$

Proof: Consider 2 cases:

Case 1: $x_1 \neq x_2$.

Just choose $x_1 \in V_1$, $x_2 \in V_2$ & $V_1 \cap V_2 = \emptyset$ such that $\varphi_1 \in [V_1, g_1]$ & $\varphi_2 \in [V_2, g_2]$

$V_1 \cap V_2 = \emptyset$

$\Rightarrow [V_1, f_1] \cap [V_2, f_2] = \emptyset.$

Case 2: $x_1 = x_2 =: x_0$

Choose $x_0 \in V \subseteq X$ open so that $f_1 \in \mathcal{F}(V)$, $f_2 \in \mathcal{F}(V)$

Then check that $[V, f_1] \cap [V, f_2] = \emptyset$ ~~the one fn~~ connected (use Identity Theorem). □

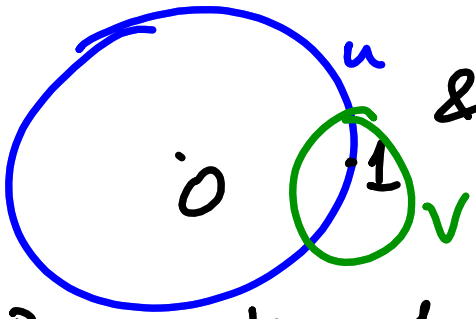
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§7. Analytic continuation

We apply the previous section to the question of analytic continuation of a holomorphic function.

Example: $X = \mathbb{C}$

$u : [0,1] \to X$ is the unit disc.



Now choose $z_0 = 1$, & V a small connected open set containing 1 but not 0.

By existence of logarithm, $\exists$ a holomorphic function $f: V \to \mathbb{C}$ so that $f(z)^2 = z$ & $f(1) = 1$

(so f is a square root of the function z.)

Question. Is there an open neighborhood V of the unit circle & a holomorphic function $f: V \to \mathbb{C}$ such that

$$\begin{cases} f(z)^2 = z & \forall\, z \in V \\ f(1) = 1 \end{cases} ?$$

(This is called analytic continuation of f to the unit circle.)

The answer is no because of monodromy. But can we do something?

The catch: if we use sheaf & sections then we can solve this analytic continuation but the resulting function does not live on $\mathbb{C}\setminus\{\}$, but on an étale space!

$\Rightarrow$ This is the idea of Riemann!

And solving this uses similar idea like what we did for lifting curves to a covering space.

$\rightarrow$ What is the sheaf $\mathcal{F}$? $\mathcal{F} = \mathcal{O}_X$

$\rightarrow$ So this is ~~st~~ exactly a lifting of curve problem. For to be able to solve this we just need to show $|\mathcal{O}_X| \to X$ is a covering map. $\square$

Remark: $|\mathcal{O}_X|$ is very big & may be connected.

Def (Analytic continuation as a Riemann surface):

Assume X = Riemann surface

$a \in X$

$\varphi \in \mathcal{O}_{X,a}$

Y a RS, $p: Y \to X$ local homeomorphism (i.e. unbranched), $f \in \mathcal{O}(Y)$, $b \in Y$, & so that $p(b) = a$ $p(f_a) = \varphi$.

Then (Y, p, f, b) is an analytic continuation

of (X, a, φ).

Def (Maximal analytic continuation):

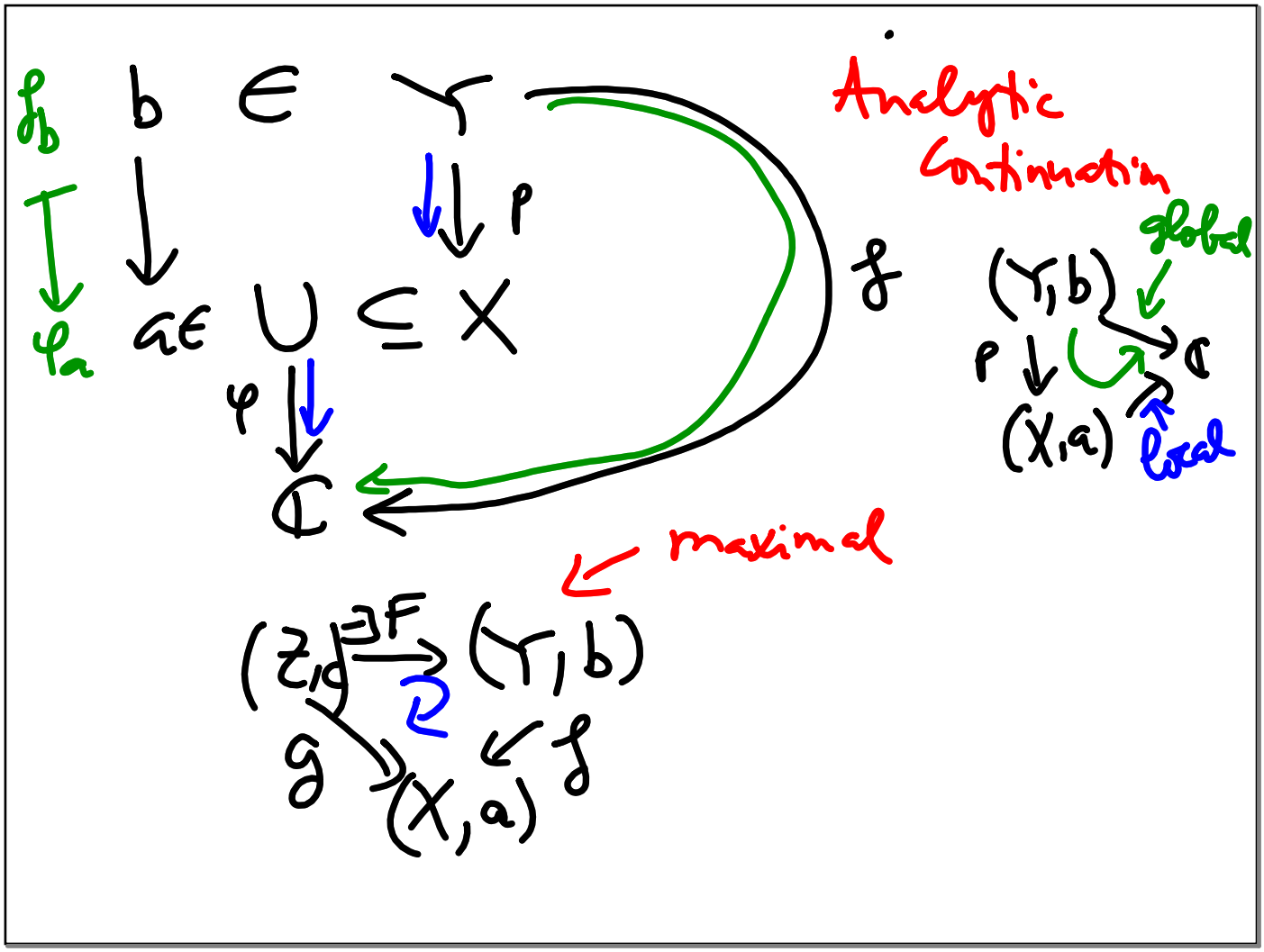
(X, a, φ) as before.

An analytic continuation (Y, P, f, b) is maximal if: whenever (Z, q, g, c) is another analytic continuation then it factors through (Y, P, f, b), i.e:

$\exists\ F : Z \to Y$ holomorphic s.t. that

$F(c) = b$ & $g = F^* f$

(i.e: $g(z) = f(F(z))\ \forall z \in Z$)


Analytic Continuation
global
local
maximal

Thm 7.8 (Existence of maximal analytic continuation):
If X is RS, $a \in X$, $\varphi_a \in \mathcal{O}_{X,a}$
$\Rightarrow$ $\exists$ maximal analytic continuation $(Y, p, *, f, b)$.

Proof:

Step 1: Construct an analytic continuation.

Y = the connected component of $|\mathcal{O}_X|$ which contains φ_a. (It exists, because $|\mathcal{O}_X|$ contains $[U, \varphi]$ for some $a \in U \subseteq X$ open.)

$p : \Upsilon \to X$ the restriction of
$p : |\mathcal{O}_X| \to X$ (in §6).

Show that p is a local homeomorphism

(for each $\varphi = (a, f_a) \in \Upsilon$
look at $[U, f] \subseteq \Upsilon$
then $p : [U, f] \to U$
$(x, f_x) \mapsto x$ is a
local homeomorph

X is RS $\Rightarrow \exists$ a topology on Υ so that
Υ is ch. RS & $p : \Upsilon \to X$ is analytic

(Basically, just ~~look at the inverse~~
identify $\tau \underline{[U, f]} \in \Upsilon$ & $\underline{U} = p([U, f])$

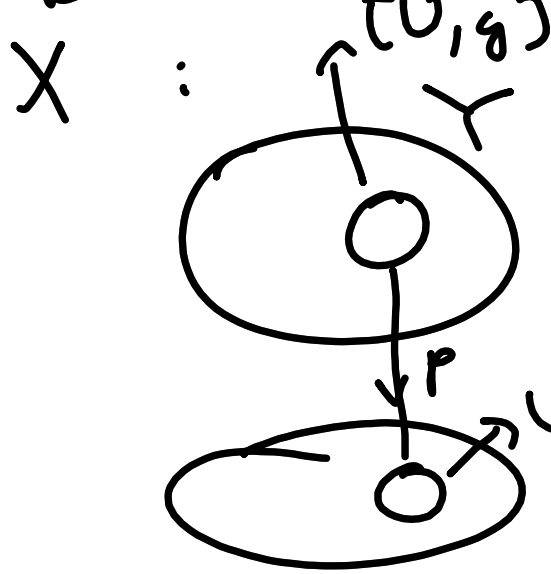
Now construct $f: Y \to \mathbb{C}$.

If $\varphi \in Y \Rightarrow \varphi = (x_0, g_{x_0})$

$g: U \subseteq X \to \mathbb{C}$ is holomorphic.

Then define $f(\varphi) := g(p(x_0))$!!

Because Y is local homeomorphic to X :



$f = g \circ p$

$\Rightarrow f$ is holomorphic.

But (locally). holomorphicity is a local property

Then $(Y, p, f, b = \varphi)$ is an analytic continuation of (X, φ, a)!!

Rmk: In this construction, Y depends on φ but not too special (Y = connected component of $|\mathcal{O}_X|$ which contain φ), p & f are independent of φ. b depends on φ (in fact $b = \varphi$).