28 October 2026
MAT 4800

* For part a) of Lemma 14.6:
Need only to prove open sets (not open covering):

$$W \subset\subset V \subset\subset U \subset\subset U^*$$

ψ cut off function:

$\psi = 1$ on W

$\psi = 0$ outside of V

$\Rightarrow$ If ξ is defined on V
$\Rightarrow \psi\xi$ is defined on U

$\psi\xi = \xi$ on W

$0 \leq \psi \leq 1$

$|\psi\xi| \leq |\xi|$ on V

$(g_{ij}) = \xi \in Z^1(\mathcal{V}, \mathcal{O})$

L^2

$\Rightarrow \exists\ (g_i) \in C^0(\mathcal{V}, \mathcal{O})$:

$g_j - g_i = g_{ij}$ on $U_i \cap U_j$

$\Rightarrow d'' g_i = d'' g_i$ (g_{ij} is holomorphic, & $d'' = \bar{\partial}$)

So $(d''g_i)|_{U_i}$: $(0,1)$ form on U_i glues together to a global $(0,1)$ form on X. Call it ω.

Now we Dolbeault lemma about solving $\bar{\partial}$ - equation for form form with compact support:

For each i: $\exists\ h_i \in \mathcal{E}(U_i^*)$ so that $d''h_i = \psi\omega$ (multiplying ψ to make sure we have a form with compact support.)

$\Rightarrow$ $h_i - g_i$ is holomorphic on W!! (only) ($d''h_i = d''g_i = \omega$)

$\Rightarrow$ the map

$$Z^1_{L^2}(\mathcal{U},\vartheta) \times \mathcal{E}^0_{L^2}(W,\vartheta) \longrightarrow \mathcal{Z}^1(\mathcal{V},\vartheta)$$

is surjective.

* Example:

$$A : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(x,y) \mapsto x+y$$

This is surjective map.

Now I want to find the points $(x,y) \in \mathbb{R}^2$ with smallest norm so that:

$A\begin{pmatrix} x \\ y \end{pmatrix} = 1.$

$\min |x| + |y|$

where $x + y = 1$

Quadratic Programming (Almost Linear programming)

Another better norm: $x^2 + y^2$

$\min x^2 + y^2$

with constraint: $x + y = 1$

* Degree 2:

$$x^2+y^2+z^2=0$$

(Fermat's surface: $x^n+y^n=z^n$) $\xi^n=1$

Over $\mathbb{C}$: $\{x^2+y^2+z^2=0\} \simeq \mathbb{P}^1$.

* Example: How to compute the genus of $x^2+y^2+z^2=0$

using Riemann-Hurwitz theorem?

(Theorem: For any compact RS, X there is a [illegible] map $f: X \to Y$)

$$X = \{[x:y:z] \in \mathbb{P}^2 : x^2+y^2+z^2=0\}$$

One simple way to construct a map from X to $\mathbb{P}^1$ is simply use the projection map:

$$[x:y:z] \mapsto [x:y].$$

Remark: this does not work for X = higher dimension!

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x+2y} = \text{no limit!!}$$

$$[X:Y:Z] \longrightarrow [X:Y]$$

$$X^2+Y^2+Z^2=0$$

Claim: degree $= 2$. $\quad X=\frac{x}{y}, Z=\frac{z}{y}$

Look at a point where $y \neq 0$:

$\Rightarrow$ the curve is:

$$\{X^2+Z^2=-1\} \longrightarrow X$$

Also we know $\chi(\mathbb{P}^1)=2-2g=2$

$\pi_1 : \mathcal{X}_1 = \{ X^2 + Z^2 + 1 = 0 \} \to \mathbb{C}$

$(X, Z) \mapsto X$

Critical points?

$\mathcal{X}_1 = \{F = 0\}$, $F = X^2 + Z^2 + 1$

$\nabla F = (2X, 2Z)$

∇F never 0 on $\mathcal{X}_1 \Rightarrow \mathcal{X}_1$ is smooth.

If $X \neq 0 \Rightarrow Z$ can be chosen to be coordinate of $\mathcal{X}_1$

$\pi_1(X, Z) = X$

Want to find critical pnts $\Rightarrow \frac{\partial \pi_1}{\partial Z} = 0$

$$X^2 + Z^2 + 1 = 0$$

$\Rightarrow$ Implicit differentiation:

$$2X\,dX + 2Z\,dZ = 0$$

$$X \neq 0 \Rightarrow dX = -\frac{2Z\,dZ}{2X} = -\frac{Z\,dZ}{X}$$

$$\frac{\partial \pi_1}{\partial Z} = \frac{dX}{dZ} = -\frac{Z}{X} \Rightarrow \frac{\partial \pi_1}{\partial Z} = 0$$

$$\Leftrightarrow Z/X = 0.$$

We are working with $X \neq 0$

$$\hookrightarrow \quad Z/X = 0 \Leftrightarrow Z = 0$$

$$\Leftrightarrow \quad X^2 + 1 = 0 \Leftrightarrow X = \pm i$$

So $\pi_*|_{\mathcal{X}_1}$ has 2 critical points

Similarly $\pi|_{\mathcal{X}_2}$ also has 2 critical points which are actually the same as for $\pi|_{\mathcal{X}_1}$.

So $\pi: \mathcal{X} \to \mathbb{P}^1$ non-const (deg 2) & 2 critical points P_1, P_2

$$\Rightarrow \quad e_1 = e_2 = 2$$

So by ~~R-H~~: Riemann-Hurwitz:

$$\chi(\mathcal{X}) = 2\chi(\mathbb{P}^1) - \underbrace{(e_1-1)}_{=2} - \underbrace{(e_2-1)}_{=2}$$

$$= 2\times 2 - 1 - 1 = 2$$

There is only 1 (upto biholomorphism) Riemann surface of $X = \mathbb{Z}$, which is $\mathbb{P}^1$.

If $\chi(\mathcal{X}) = 2 \Rightarrow g(\mathcal{X}) = 0$. □

Actually degree - genus formula (for a curve in $\mathbb{P}^2$) can be proven using Riemann - Hurwitz theorem.

Belyi's theorem.