MAT 4800

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§7 (Cont.) ⟸ Connected RS

* $f: X \to Y$, holomorphic, non-constant

$x_0 \in X$

$\Rightarrow$ local model of f at x_0 is

$$z \mapsto z^k,$$

$k \geq 1$ multiplicity of f at x_0, also denoted e_{x_0}.

$k > 1 \Leftrightarrow f'(x_0) = 0 \Rightarrow$
only a discrete set.

If X is compact RS $\Rightarrow$
a discrete set on X is finite.

* Riemann – Hurwitz theorem (compact ← RS)
$\Rightarrow$ if $\exists\ f: X \to Y$
holomorphic, non-constant,
$\Rightarrow \chi(X) \leq N \cdot \chi(Y)$
for some $N \geq 1$.

Example: There is no non-constant holomorphic map from $\mathbb{P}^1 \to E$ (elliptic curve)

$$\chi(\mathbb{P}^1) = 2 - 2g_{\mathbb{P}^1} = 2$$

$$\chi(E) = 2 - 2g_E = 0$$

$\nexists\ N \geq 1$ so that $2 \leq N \cdot 0 = 0$

* Example: ~~$\chi(\mathbb{P}^1) = 2$~~ $g(\mathbb{P}^1) = 0 < 1$

There are many holomorphic maps no-constant & not automorphism from $\mathbb{P}^1 \to \mathbb{P}^1$

(Example: $[x:y] \mapsto [x^n : y^n]$ $n \geq 2$

* Example: There are many holomorphic, non-constant, non-automorphic maps $E \to E$ for elliptic curves.

For every $N \geq 2$, define the map

$$E \to E, \quad z \mapsto N \cdot z = \underbrace{z + \dots + z}_{N \text{ times}}$$

where $+$ is the group operation on E.

(E is a group.)

Explain how to define "$+$" on E:

Assume E has the equation (always possible by using Weierstrass equation):

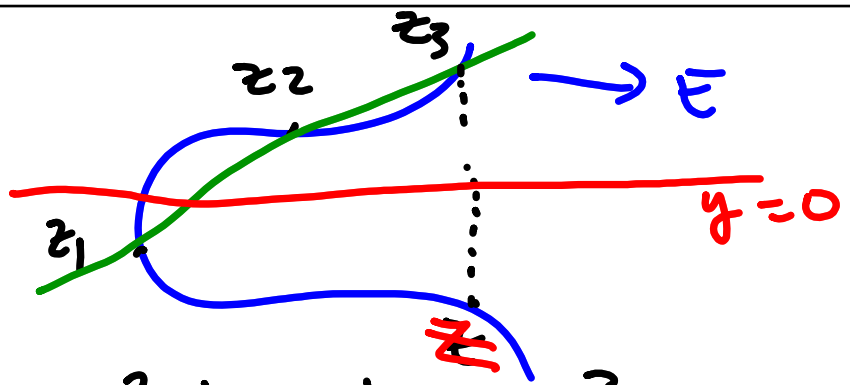
$y^2 = ax^3 + bx + c$, $a \neq 0$

& all roots of $ax^3 + bx + c$ ~~has no double~~ are simple.

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ be 2 points on E. We want to define $z = z_1 + z_2$ so that $z \in E$ & this $+$ is a group operation.

2 cases:

Case $z_1 \neq z_2$:



$E \subseteq \mathbb{C}^2$ has degree 3

Let $L =$ be the line going through z_1 & z_2

$\Rightarrow L \cap E$ has 3 points.

Let z be the reflection of z_3 via the line $y=0$. We define: $z_1 + z_2 := z$.

Case if $z_1 = z_2$:

Then the line through z_1 & z_2 becomes the tangent line L of E at z_1.

Then we do as above.

HW: This is a geometric description.

Work out the algebraic expressions for the coordinate of $z_1 + z_2$, & see that it is some simple rational function of (x_1, y_1) & (x_2, y_2). However, if we iterate these: i.e compute coordinates of

z, $2z = z+z$, $3z = z+2z$, ...

then the formulas become quickly complicated

that's a reason why elliptic curves are used in cryptography. (They do it on finite fields,

↳ loved in computer

* The idea of the proof of RH theorem: If we take out the preimage of critical values (i.e. those with local degree > 1) then the map is a covering. Euler characteristic

behaves well with the covering maps.
(i.e $\chi(X)$ w.s N. $\chi(Y)$).
Now the term $-\sum_{x \in X} (\ell_x - 1)$ is to take care of what happens at the critical points.

Another special property of Euler's characteristic is that it is "additive":

$$\chi(X_1 \amalg X_2) = \chi(X_1) + \chi(X_2) \text{ !!}$$

Also Euler characteristic is a topological invariant !!

* Continue of Theorem 7.8: We constructed (Υ, ρ, f, b). Now we show that it is maximal.

Main idea for maximality: since Υ is constructed from $|\mathcal{O}_X|$, so tautologically it should be maximal, because $|\mathcal{O}_X|$ contains all possible germs of holomorphic functions on X.

$$(\Upsilon, \rho, f, b)$$
$$\downarrow$$
$$(Z, q, g, c) \to (X, \varphi, a)$$

Let $\gamma \in \mathcal{Z}$ & we need to define $F(\gamma) = \eta \in Y$.

Because Y contains all germs of functions which are possible analytic continuation of φ over all curves) & $(\mathcal{Z}, q, g, c)$ is such one $\Rightarrow$ $\exists$ unique (because germ of function is unique) $\eta \in Y$ s.t. that

$$q^*(\underbrace{\rho_\gamma(g)}_{\text{the germ of } g \text{ at } q_\gamma}) = \eta. \quad \square$$

* In §8: In the case of analytic continuation of algebraic functions, we obtain very nice results.

Y is then also algebraically dependent on X.

* <u>The meaning of Theorem 8.9</u>:

$X = RS$

& $P(T) = T^n + c_1 T^{n-1} + \cdots + c_{n-1} T + c_n$,

$c_1, \ldots, c_n \in k(X)$ = rational functions on X

$(\Rightarrow)$ $c_1, \ldots, c_n : X \to \mathbb{P}^1$ holomorphic.

$\exists$ Y RS

& $f: Y \to \mathbb{P}^1$

~~so that~~

& $\pi: Y \to X$ degree n = degree p

So that $f(y)^n + c_1(\pi(y)) f(y)^{n-1}$

$+ c_2(\pi(y)) f(y)^{n-2} + \dots$

$+ c_{n-1}(\pi(y)) f(y) + c_n(\pi(y))$

$\equiv 0 \quad \forall y \in Y.$

Another way to say is if we consider

$(\pi, f): Y \to X \times \mathbb{P}^1$

then for $(x,t) = (\pi(y), f(y))$

we have:

$$t^n + c_1(x)t^{n-1} + \dots + c_{n-1}(x)t + c_n(x) = 0 \quad (*)$$

i.e. the image of Y is the curve in $X \times \mathbb{P}^1$ defined by $(*)$.

In general Y ~~is not~~ cannot be chosen to be X but a RS surface over X ($Y \subseteq X \times \mathbb{P}^1$).

Corollary. If X is algebraic (i.e an open dense subset of a compact RS, then so is Y).

Example: $X = \mathbb{C}$ ← algebraic.

(Revisit examples from Gamelin's book Complex analysis, MAT 2410).

& we want to define $\sqrt{x^3 - x + 1}$

Then look at $P(t) = t^2 - (x^3 - x + 1)$

$\Rightarrow Y \subset X \times \mathbb{P}^1$ defined by

$\{(x,t) : t^2 - (x^3 - x + 1) = 0\}$

which is an elliptic curve !!

Look at the projection map

$\pi : Y \to X$

$(x,t) \mapsto x$ $\Rightarrow$ it has degree 2!!

HW. Explain why we only need to consider (in this example) $Y \subseteq X \times \mathbb{C}$ & not $X \times \mathbb{P}^1$ (i.e. the point $\infty \in \mathbb{P}^1$ is not needed, because of Fundamental Theorem of Algebra.)

What if you want $\sqrt{\frac{x^3-1}{x}}$

* Main idea for the proof of Theorem 8.9:
prove it locally near points where $c_1, c_2, \ldots, c_{n-1}, c_n$ are holomorphic.

At such a point x_0 $\Rightarrow$ show the existence of f & π in a neighborhood of x_0 which satisfy the equation:

$$t^n + c_1(x)t^{n-1} + \cdots + c_{n-1}(x)t + c_n(x) \equiv 0$$

for $(x,t) = (f(y), \pi(y))$.

Then show that if x_0 is a singular point of $(c_1, \ldots, c_n)$ (so it must be a pole, because $c_1, \ldots, c_n$ are rational functions)

$\Rightarrow$ π is removable at x_0
& f is not either removable or pole at x_0.

How do we show existence of f & π locally?

For each x_0 for which $c_1(x_0), \dots, c_n(x_0)$ $\neq \infty$ $\Rightarrow \exists$ a neighborhood U of x_0 s.t. that we have also the same for all $x \in U$.

Then define $Y|_U$ locally as

$$Y|_U := \{ (x,t) \in U \times \mathbb{C} : t^n + c_1(x) t^{n-1} + \dots + c_{n-1}(x) t + c_n(x) = 0 \}$$

$Y|_U$ is a RS!! What is the map π, & the map f?

$\pi|_{Y|_U} : y=(x,t) \longmapsto x \in U$

(π is degree n because of the Fundamental theorem of Algebra)

$f|_{Y|_U} : y=(x,t) \longmapsto t \in \mathbb{C}$!!

These maps are very simple!!

If U & V are 2 open sets so that ~~U∩V~~ $(c_1,\dots,c_n)$ are holomorphic on $U \cup V \Rightarrow Y|_U$ & $Y|_V$ can be glued.

This way we get a RS on Y over $X \setminus \{x : (c_1,\dots,c_n) \text{ is not holomorphic}\}$.

& the trick is to show that ~~[illegible]~~ π & f can

be extended over this singular set.

Now we have:

$$\Upsilon^0 = \{ (x,t) \in (X \setminus \text{singularity}) \times \mathbb{C} : t^n + c_1(x)t^{n-1} + \dots + c_{n-1}(x)t + c_n(x) = 0 \}$$

& Υ is just the closure of Υ^0 in $X \times \mathbb{P}^1$!! □

Question: Why if at least one of $c_1(x_0), \dots, c_n(x_0) = \infty$ then solutions t of $P(t,x) = 0$ cannot be essential singularities?

One remark: The above sketch only works if the curve Y is smooth (we want a RS). In the general case, look at the accompanied note (from 2023), where to deal with the general case, then we need a result concerning a covering map of finite degree from $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ to itself.

HW: Show that if for $P(t,x) = t^n + c_1(x)t^{n-1} + \dots + c_{n-1}(x)t + c_n(x)$, & $Y = \{(t,x) \in \mathbb{C}^2 : P(t,x) = 0\}$, then if (t_0, x_0) is such that $t = t_0$ is a simple root of the polynomial $P(t, x_0)$ then Y is smooth at (t_0, x_0). This explains the role of the discriminant Δ of $P(t,x)$ in the complete proof. $\square$

Hint for HW:

Assume that we have essential singularity at x_0 for t. Then there are infinitely many limits of t when $x \to x_0$. Say t^* be such a limit, then for example if we look at residue of $c_1(x), \dots, c_n(x)$ at $x_0 \Rightarrow$ then if we write:

$$c_j(x) = \frac{\alpha_j}{(x - x_0)^{n_j}} + \text{smaller order}$$

& let $n = \max_j n_j$

then we must have:

$$\sum_{j:\, n_j = n} \frac{\alpha_j}{(x - x_0)^n} \cdot (t^*)^{n-j}$$

$$\equiv 0 \text{ identically in } x.$$

But if we have essential singularity for $t \Rightarrow t^*$ can be dense in $\mathbb{C}$,

& show that this yields a contradiction.