Math 4800
2 September 2026

* Proof that if $f: X \to Y$, compact RS (connected), non-constant holomorphic maps then $\exists\ d \geq 1$ = the degree of f so that $\forall\ y_0 \in Y$, $\#\, f^{-1}(y_0) = d$ (counted multiplicities).

Not true if X or Y is noncompact.

E.g. $X = Y = \mathbb{C}$, $f(z) = e^z$

$f^{-1}(0) = \infty$

$\Rightarrow$ First, because f has open image (Open Mapping) & $f(X)$ closed (because X is compact) $\Rightarrow f(X) = Y$.

$\Rightarrow$ Local model: $\forall x_0 \in X$, $\exists x_0 \in U_0 \subseteq X$ s.t. $f: U \to Y$ has the model $z \mapsto z^k$ ($k \geq 1$)

Now look at: $f: B(0,1) \to B(0,1)$, $z \mapsto z^k$

Then every point in $B(0,1)$ has the same number of preimages (k).

If $c \in B(0,1)$ $(|c|<1)$

$f^{-1}(c) = \{ z \in B(0,1) : z^k = c \}$

If $c \neq 0$ then $\# f^{-1}(c) = k$

(Using k-th roots of unity)

$z_0^k = c$

$z^k = c \iff \left(\frac{z}{z_0}\right)^k = 1 \iff \frac{z}{z_0}$ is a k-th root of 1

$\Rightarrow X$ is compact so we can cover X by a finite number of such open sets.

* $\mathbb{P}^2$:

Recall $\mathbb{P}^1$ = Riemann sphere

$=$

$\mathbb{C}_z \cup \mathbb{C}_w$: $w = \frac{1}{z}$ for $w, z \neq 0$

In other words: $\mathbb{P}^1$ = ~~closed~~ the set $\{(z,w) \in \mathbb{C}^2$:

* $z, w \neq 0$ & $z = \frac{1}{w}$
* $(z=0, w \neq 0) \sim (0,1)$
* $(z \neq 0, w = 1) \sim (1,0)\}$

$[z:w] \in$

$z \neq 0$ or $w \neq 0$

* $\mathbb{P}^2$:
= the set of complex lines in $\mathbb{C}^3$ which go through $(0,0,0)$

Each such line is represented by
$ax+by+cz=0$
& $(a,b,c) \neq (0,0,0)$

Represent it by $[a:b:c]$

Then $\mathbb{P}^2$ is covered by 3 open sets (similar to $\mathbb{P}^1$), each is isomorphic to $\mathbb{C}^2$.

If $[a:b:c] \in \mathbb{P}^2$ & $c \neq 0$

$\Rightarrow [a:b:c] = [\frac{a}{c} : \frac{b}{c} : 1]$

$\| \quad \|$

$X_3 \quad Y_3$

$U_3 = \{[a:b:c] \in \mathbb{P}^2 : c \neq 0\}$

$\simeq \mathbb{C}^2$

(X_3, Y_3)

$[a:b:c] \longrightarrow X_3 = \frac{a}{c}, \; Y_3 = \frac{b}{c}$

$[X_3 : Y_3 : 1] \longleftarrow (X_3, Y_3)$

Similarly: if $a \neq 0$:

$$V_1 = \{ [a:b:c] : a \neq 0 \}$$

$$\approx \mathbb{C}^2_{Y_1, Z_1}$$

$$[a:b:c] \to Y_1 = b/a, Z_1 = c/a$$

$$[1:Y_1:Z_1] \leftarrow$$

$$U_2 = \{[a:b:c] : b \neq 0\}$$

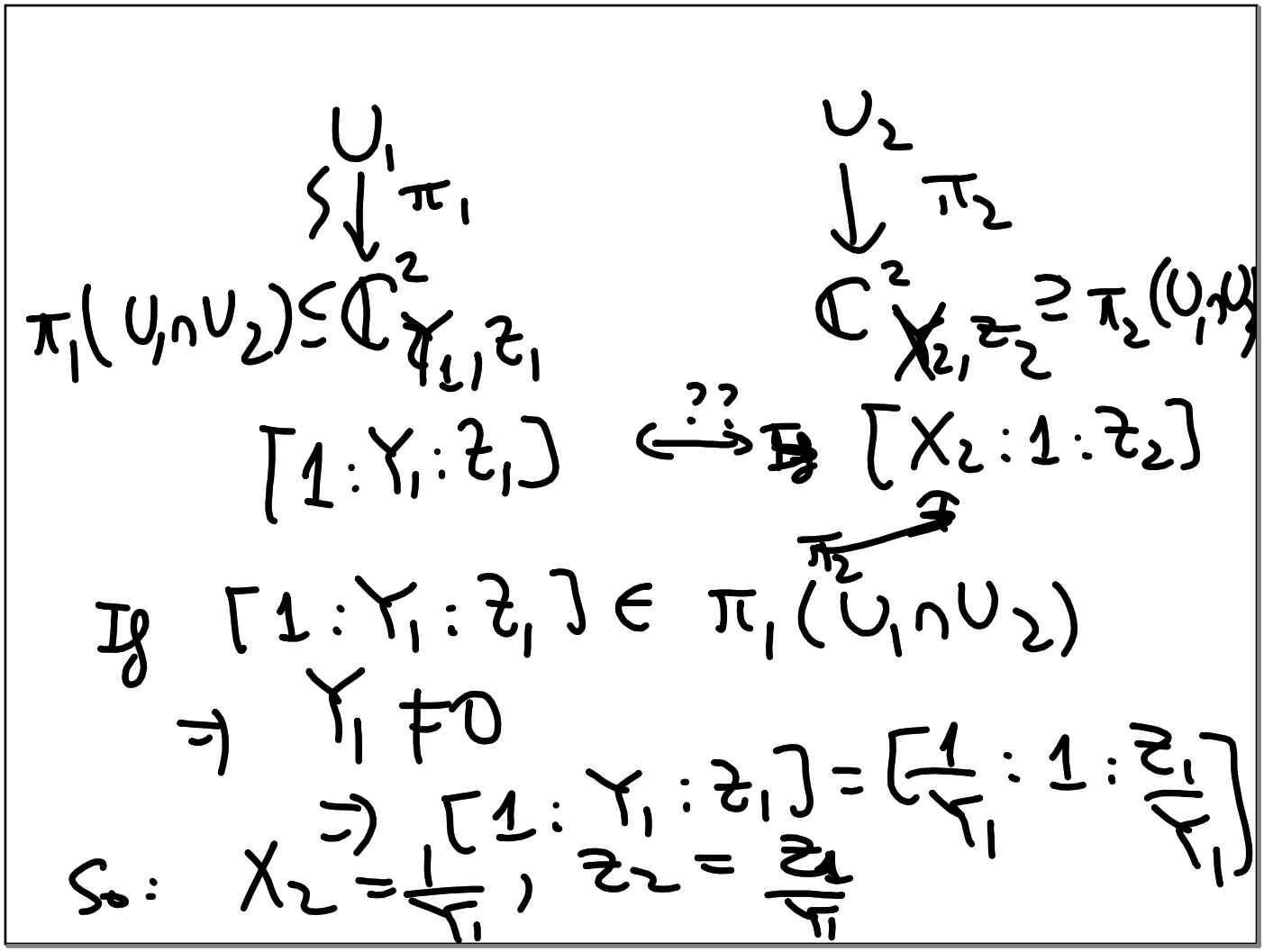
$$\approx \mathbb{C}^2_{X_2, Z_2}$$

$$[a:b:c] \mapsto X_2 = \frac{a}{b}, Z_2 = \frac{c}{b}$$

$$[X_2:1:Z_2] \longleftarrow$$

$$U_1 \cap U_2 = \{[a:b:c] \in \mathbb{P}^2 : a, b \neq 0\}$$

What is the transition function for $U_1 \cap U_2$?


U_1
π_1
$\pi_1(U_1 \cap U_2) \subseteq \mathbb{C}^2_{Y_1, Z_1}$
$[1 : Y_1 : Z_1]$
U_2
π_2
$\mathbb{C}^2_{X_2, Z_2} \supseteq \pi_2(U_1 \cap U_2)$
$[X_2 : 1 : Z_2]$
If $[1 : Y_1 : Z_1] \in \pi_1(U_1 \cap U_2)$
$\Rightarrow Y_1 \neq 0$
$\Rightarrow [1 : Y_1 : Z_1] = \left[\frac{1}{Y_1} : 1 : \frac{Z_1}{Y_1}\right]$
So: $X_2 = \frac{1}{Y_1}$, $Z_2 = \frac{Z_1}{Y_1}$

* Weierstrass theorem:

$E = \mathbb{C}/\Lambda$ is elliptic curve

$\Rightarrow \exists \quad f : \mathbb{C} \to \mathbb{P}^1$ holomorphic & doubly periodic (with period in Λ)

such that

$$\begin{array}{ccc} \mathbb{C} & \to & \mathbb{P}^2 \\ \cup & & \cup \\ z & \mapsto & [f(z) : f'(z) : 1] \end{array}$$

holomorphic

is an embedding of E in $\mathbb{P}^2$.

(f is holomorphic & injective).