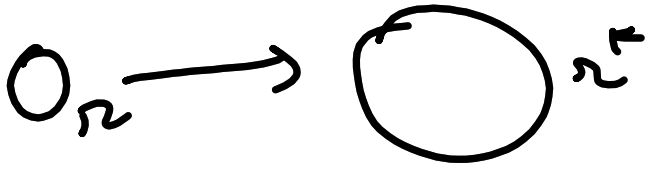
MAT 4800
31 August 2026

For open Mapping theorem, another proof: Locally, the map is

$$z \to z^k, \quad k \geq 1$$



Riemann removable theorem:

$g: B(a,r) \to \mathbb{C}$ holomorphic

But now restrict it to $B(a,r)\setminus\{a\}$:

$f: B(a,r)\setminus\{a\} \to \mathbb{C}$

$f := g$

$g(z) = g\, z^k, \quad z \in B(0,1)$

$f = g|_{B(0,1)\setminus\{0\}}$

Poles:

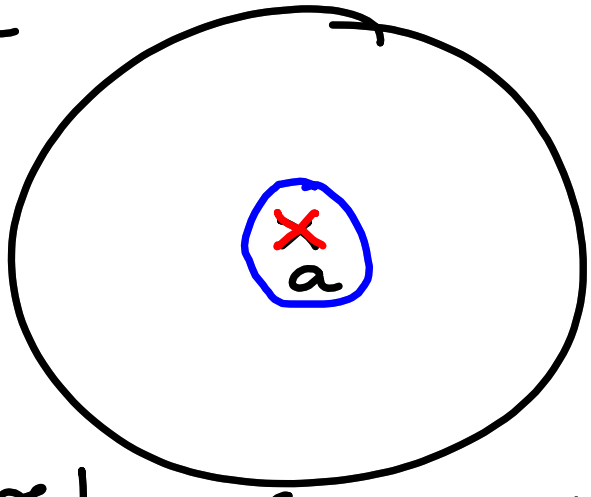
$f(z) = \frac{1}{z} : B(0,1)\setminus\{0\} \to \mathbb{C}$

$|f(z)| = \frac{1}{|z|} \to \infty$ if $z \to 0$

Essential singularities:

if c is not in the closure of image

$\Rightarrow \frac{1}{f(z)-c}$ is in case 1 ....



Casorati-Weierstrass theorem:

Image of the blue disk is dense in $\mathbb{C}$.

Get a contradiction.

What kind of singularity for $z=\infty$ of the function e^z?

The question means that if we put $z = \frac{1}{w}$, so $z=\infty \Leftrightarrow w=0$

$f(w) = e^{\frac{1}{w}}$, & what is the type of singularity at $w=0$?

Case 1. $\lim_{w\to 0} f(w)$ exists?

$$w_n = \frac{1}{2n\pi i + \frac{\pi}{2}} \longrightarrow 0$$

$$f(w_n) = e^{\frac{1}{w_n}} = e^{\frac{\pi}{2} + 2n\pi i} = e^{\pi/2}$$

$$\hat{w}_n = \frac{1}{2n\pi i + \frac{\pi}{3}} \longrightarrow 0$$

$$f(\hat{w}_n) = e^{\frac{1}{\hat{w}_n}} = e^{\pi/3}$$

So $w_n, \hat{w}_n \to 0$ but $f(w_n), f(\hat{w}_n)$ not converge to the same point (like f like on infinity $\Rightarrow$ **Essential singularity**

If $U \overset{\neq \emptyset}{\subseteq} \mathbb{C}$, then there is a non-constant holomorphic function:

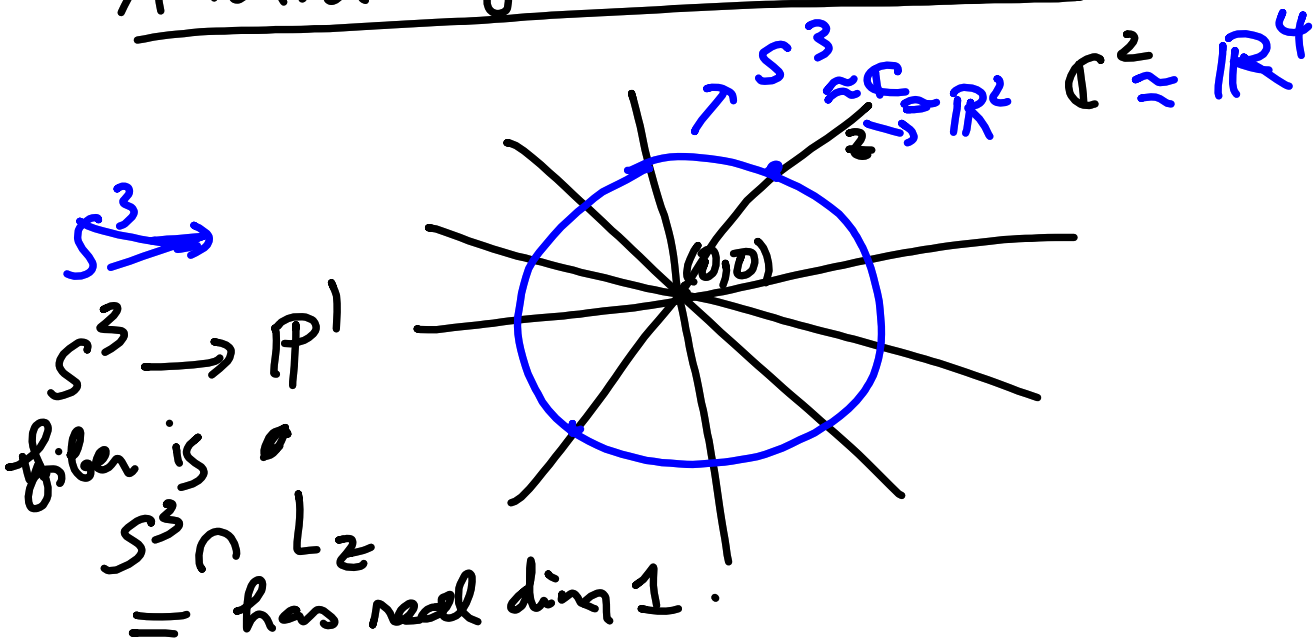
$$f: U \to \mathbb{C}$$

$\cong$ identity map

If $\mathbb{P}^1$ were $\subseteq \mathbb{C}$, then we get a contradiction.

($\mathbb{P}^1$ is compact)

Another way to describe $\mathbb{P}^1$:



$S^3 \to \mathbb{P}^1$

fiber is $\bullet$

$S^3 \cap L_z$

= has real dim 1.

* $X \subseteq \mathbb{C}^2$:

$= \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$

where $f: \mathbb{C}^2 \to \mathbb{C}$ analytic.

(e.g.: $f(x,y) = \sin(x) + e^y - 1$

$f: B((a,b),r) \to \mathbb{C}$ is holomorphic

if it in $\mathbb{C}^2$ has an absolutely convergent power series representation:

$a = (x_0, y_0) \in \mathbb{C}^2$

$$f(x,y) = \sum_{m,n=0}^{\infty} c_{m,n} (x-x_0)^m (y-y_0)^n$$

absolutely convergent.

$f: U \text{ open} \subseteq \mathbb{C}^2 \to \mathbb{C}$

is holomorphic if $\forall a=(x_0,y_0) \in U$,

$\exists\ r > 0$ so that $B(a,r) \subseteq U$

$f: B(a,r) \to U$ is holomorphi

$$X = \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$$

is a Riemann surface (here $f: \mathbb{C}^2 \to \mathbb{C}$ is holomorphic) iff

X is a real manifold.

(X is a 2-dim real manifold $\subseteq \mathbb{C}^2 \cong \mathbb{R}^4$)

* Why called Riemann surface?

For example we want to define the "square root function":

$$w = \sqrt{z}.$$

If $z \neq 0$ $\Rightarrow$ we have 2 square roots & a function has only one value $\Rightarrow$ need to choose one of these 2 square roots.

If $0 \notin U$ open $\subseteq \mathbb{C}$, then by the existence of logarithm: $\exists\ F: U \to \mathbb{C}$ holomorphic function so that

$$z = (F(z))^2 \quad \forall z \in U$$

So $w = F(z)$ is square root.

Question: if $0 \in U$, can there be a holomorphic function $F: U \to \mathbb{C}$ so that $z = (F(z))^2 \quad \forall z \in U$?

Monodromy idea to show no = exists.

For example, choose $U = B(0,2)$:

$F(1) = 1$.

holomorphic

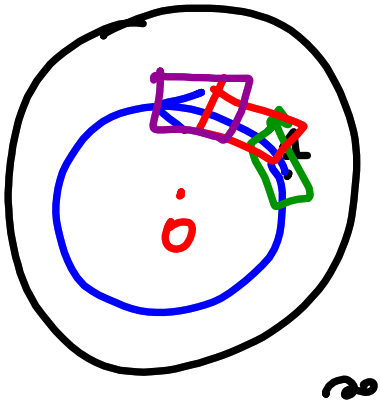
Only 1 function

F: Green rectangle $\to \mathbb{C}$

so that $F(1) = 1$,

$F(z)^2 = z$.

Go around the point 0, and observe how the function value changes.



In particular: for $z = e^{i\theta} \in$ on the unit circle: & θ close to 0 (z close to 1. $\Rightarrow$

$F(e^{i\theta}) = e^{i\theta/2}$.

$\Rightarrow$ Show that when you go to the red box, you should have the same formula.

Continue a finite number of times (because the unit circle is compact), obtain that at $z = 1$:

If we write $1 = e^{i \cdot 0}$ $\Rightarrow$ $F(z=1) = e^{i \cdot 0/2} = 1$

If we write $1 = e^{i \cdot 2\pi}$ $\Rightarrow$ $F(z=1) = e^{i \cdot 2\pi/2} = e^{i\pi} = -1$

So we obtain different values for $z=1$, a contradiction.

Riemann solved this by saying that: when θ goes from 0 to 2π, you do

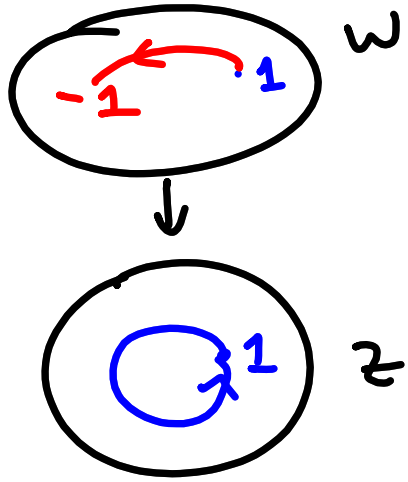
not comeback ($F(z)$) to $F(z) = 1$ but $F(z) = -1$.

Which means that the domain of this "square root" is not $\mathbb{C}$!

Actually: it is $\in \mathbb{C}^2$

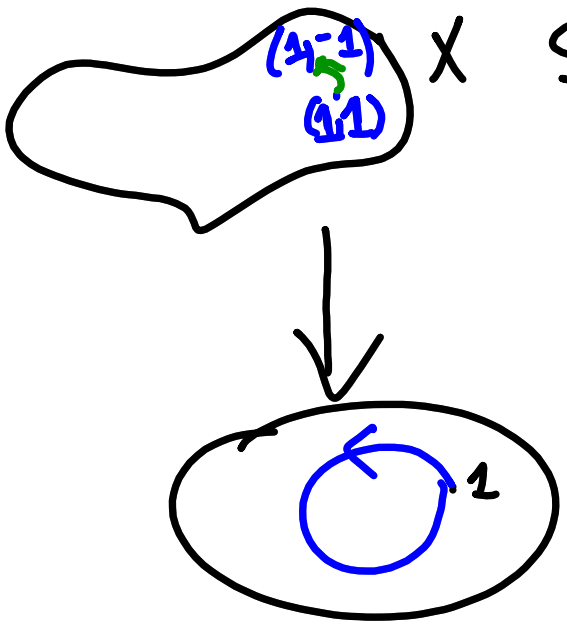
$$X = \{(z, w) : w = z^2\}$$

If we think about $F(z) = \sqrt{z} : \mathbb{C}_{\neq 0} \to \mathbb{C}_w$



Contradiction!!

We have



$X \subseteq \mathbb{C}^2_{z,w}$

no contradiction, because we have 2 coordinates!!

$\{w = z^2\} \subseteq \mathbb{C}^2$

$\mathbb{C}_z$ is Riemann surface for $\sqrt{z}$.

* For the proof of theorem:

$$X = \{(z_1, z_2) \in \mathbb{C}^2 : f(z_1, z_2) = 0\}$$

so that $\forall (z_1^*, z_2^*) \in \mathbb{C}^2$:

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}\right)(z_1^*, z_2^*)$$

$$\neq (0,0)$$

$\Rightarrow$ X is RS.

Another way. Near (z_1^*, z_2^*):

Taylor's expansion: (if $(z_1, z_2) \in X$)

$$0 = f(z_1, z_2)$$
$$= f(z_1^*, z_2^*) + \frac{\partial f}{\partial z_1}(z_1^*, z_2^*) \cdot (z_1 - z_1^*)$$
$$0 + \frac{\partial f}{\partial z_2}(z_1^*, z_2^*) \cdot (z_2 - z_2^*)$$
$$+ \dots$$

If $\frac{\partial f}{\partial z_1}(z_1^*, z_2^*) \neq 0$, we can solve:

$$z_1 - z_1^* = -\frac{\frac{\partial f}{\partial z_2}}{\frac{\partial f}{\partial z_1}}(z_1^*, z_2^*)\cdot(z_2 - z_2^*) + \dots$$

So we can express z_1 as a "holomorphic" function of z_2. So in this small set $X \cap \{|z_1 - z_1^*|, |z_2 - z_2^*| < \varepsilon\}$ we can define a map $X \to z_2 \in \mathbb{C}$, $(z_1, z_2) \mapsto$ biholomorphic.

HW: $X = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^2 - z_2^3 - 1 = 0\}$

Show that X is orientable!

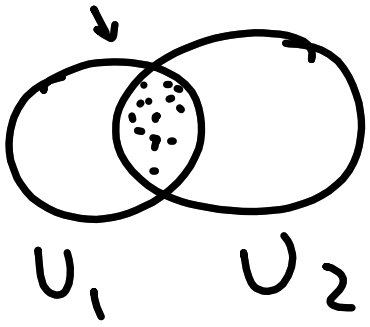
(Idea: to use Cauchy-Riemann equation!!)

We can work locally. Then X has one the 2 forms:

$X_1 = \{(z_1, z_2) : z_1 - f(z_2) = 0\}$

or $X_2 = \{(z_1, z_2) : z_2 - g(z_1) = 0\}$

Rmk: Every complex manifold (RS = dim 1) is orientable, because of Cauchy-Riemann equation.


U_1
U_2