MAT 4800

5 October 2026

* How to compute the genus of a RS to a polynomial $p(T, x)$?

Example: genus of RS to $p(T|x) = T^2 - x^2$, $x \in \mathbb{P}^1$.

Let $\mathcal{X}$ be the RS. We worked out that we need to add points since $\{T^2 - X^2 = 0\}$ is singular.

Work first on $\mathbb{C}$: then:

$\{T^2 - X^2 = 0 : (X, T) \in \mathbb{C}^2\}$ has a singular point at $(X = T = 0)$.

Recall: $\{\Delta = 0\}$ ↪ discriminant = the set of X where $P(T, X)$ has a root of multiplicity ≥ 2.

$Y' = \pi^{-1}(X \setminus \Delta) \to X \setminus \Delta$

is a covering map of degree ~~at least~~ N

$= \deg_T P(T, x)$

To get the RS for $P(T, x)$ need to add a discrete set of points to Y'.

So what we do is:

For any $x_0 \in \Delta \Rightarrow$ choose a small neighborhood $U_{x_0} \subseteq X$ & let

$\pi^{-1}(U_{x_0} \setminus \{x_0\}) =$ disjoint union of the punctured discs $V_1, \dots, V_{K(x_0)}$

$\pi|_{V_j}$ has degree n_j &

$$\sum n_j = \deg_T P = N.$$

For each V_i we add a point $u_i^{(x_0)}$

$$\Rightarrow Y = Y' \cup \bigcup_{x_0 \in \Delta} (u_1^{(x_0)} \dots, u_{k(x_0)}^{(x_0)})$$

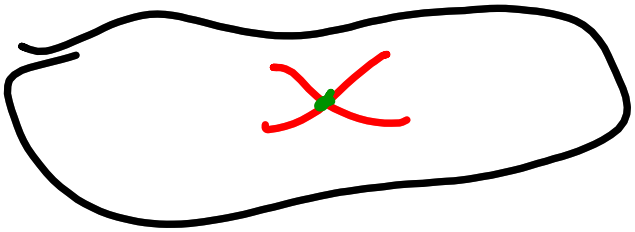
Δ: discrete set (except very special cases like $P(T,x) = T^2$. So $\Delta = \mathbb{C}$!!)

In our case $Y' = \{$ the smooth part of $\{T^2 - X^2 = 0 : x \in \mathbb{C}\}$ &

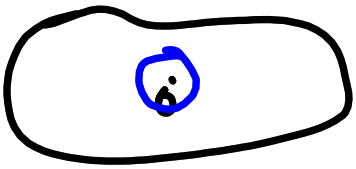
We discussed that we need to add

2 parts to Y' to get RS Y for $f(T,x) = T^2 - x^2$ over $\mathbb{C}$ $x \in \mathbb{C}$.

So here is the picture:

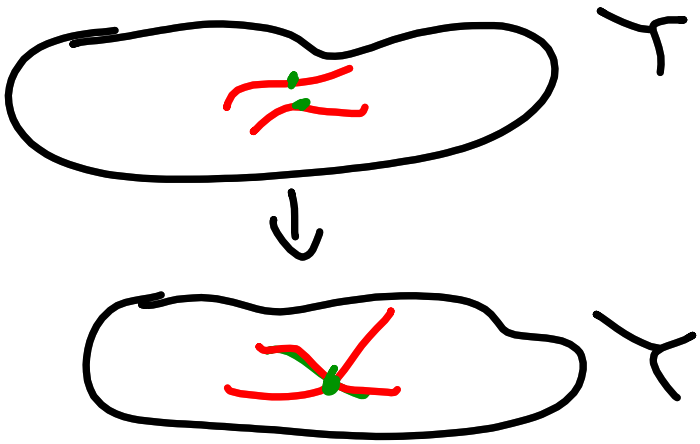


$Y' \subseteq \mathbb{C}^2$



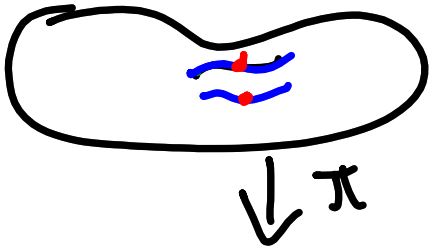
$\mathbb{C} = X$

What we did was Y = desingularisation of Y'.



Question: How to compute the genus of Y? (Only 1 singular point, the point we already discussed! → Need to check.)

First, compute the Euler's characteristic
(like in the proof of Riemann-Hurwitz
theorem)



π: degree 2

~~& π~~

~~smooth & connected~~ ← inherent

$Y \simeq \mathbb{R}S \Rightarrow$ Y might be
$X \simeq \mathbb{P}^1$ compact $\Rightarrow$ Y also compact.

$\Rightarrow 2 - 2g(Y) = \chi(Y)$

(g: genus; χ: Euler's characteristic)

$\pi : Y \to X$ degree 2

$\pi : \pi^{-1}(X \setminus U_0) \to X \setminus U_0$

is covering map

$\pi^{-1}(U_0) = V_1 \sqcup V_2$

V_1: closed disc, V_2: closed disc

$\pi^{-1}(U_0)$: closed disc

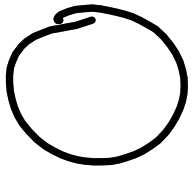
$\Rightarrow \chi(Y) = \chi(\pi^{-1}(X \setminus U_0)) + \chi(\pi^{-1}(U_0)) = \chi(\pi^{-1}(X \setminus U_0)) + \chi(V_1) + \chi(V_2) = 2 + 1 + 1 = 4.$

Now we check another way.

$X = \mathbb{P}^1$, U_0 = closed disc

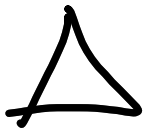
$\Rightarrow \chi(X \setminus U_0) = \chi(X) - \chi(U_0)$

$= 2 - 1 = 1$



U_0

$\sim$ homotopic



← 3 vertices
3 edges
1 surface

$\Rightarrow \chi(U_0) = 3 - 3 + 1 = 1$

$\pi: \pi^{-1}(X \setminus U_0) \to X \setminus U_0$ covering map of degree $= 2$

$\Rightarrow \chi(\pi^{-1}(X \setminus U_0)) = 2\,\chi(X \setminus U_0)$

$= 2 \times 2 = 4.$

Directly apply Riemann-Hurwitz theorem

$\pi: Y \to X$

degree 2,

also $\pi^{-1}(U_0) \cong$ disjoint union of 2 closed disc.

$\Rightarrow$ Every point has local degree 2

$\Rightarrow \quad \chi(Y) = 2\chi(X) = 4$

(Same as calculated the other way).

If Y were connected then this would be a contradiction to the formula $2-2g(Y) = \chi(Y)$ / $g(Y) \geq 0$).

But there is no contradiction since actually Y is the disjoint union of 2 $\mathbb{P}^1$, each $\mathbb{P}^1$ has $\chi = 2$.

In fact $Y' = \{(T,x) \in \mathbb{P}^1 \times \mathbb{P}^1 : T^2 - x^2 = (T-x)(T+x) = 0\}$

$=$ smooth part of $\{(T,x) \in \mathbb{P}^1 \times \mathbb{P}^1 : T = x\} \cup \{(T,x) \in \mathbb{P}^1 \times \mathbb{P}^1 : T = -x\}$.

* Why residue is independent of choice of coordinates? Let $\omega = c\frac{dz}{z}$ & let's say $z = \varphi(y)$, φ is local isomorphism

Then in y - coordinate: $\left(\begin{array}{l}\varphi(0)=0\\ \varphi'(0)\neq 0\end{array}\right.$

$$\omega = c\cdot\frac{dz}{z}$$

$$= c\cdot\frac{d(\varphi(y))}{\varphi(y)}$$

$$= c\cdot\frac{\varphi'(y)\,dy}{\varphi(y)} \sim c\,\frac{(\varphi'(0)+O(y))\,dy}{\varphi'(0)\,y+O(y^2)}$$

$$\sim c\cdot\frac{dy}{y} + \text{higher order (when } y \text{ is small!)}$$

* $z = x+iy$, $\bar{z} = x-iy$

$\Rightarrow dz = dx+i\,dy$, $d\bar{z} = dx-i\,dy$

Differential forms.

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$ are vector fields

which are dual to differential forms.

$$\frac{d}{dx}\left(\frac{\partial}{\partial x}\right) = 1, \quad \frac{d}{dx}\left(\frac{\partial}{\partial y}\right) = 0$$

$$\frac{d}{dz}\left(\frac{\partial}{\partial z}\right) = 1, \quad \frac{d}{dz}\left(\frac{\partial}{\partial \bar{z}}\right) = 0$$

So we can use the relation between

$dx, dy, dz, d\bar{z}$ to get relations

between $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}$.

* Remark: To find critical point of a holomorphic map $\pi: X \to Y$,

X & Y RS.

For each x_0 in X, find local coordinate z of X near x_0. ($z=0$ at x_0)

Then compute $\frac{d\pi}{dz}$ at x_0.

If $\frac{d\pi}{dz}(x_0) = 0 \Rightarrow x_0$ is a critical point of π.

To compute $\frac{d\pi}{dz} \Rightarrow$ need to use Implicit Differentiation.

* On meromorphic 1-form on a connected RS X:

ω is a meromorphic 1-form on a RS X $\Rightarrow$ then it is uniquely defined on any dense open subset of X.

It means: We can choose U any dense open subset of X, & compute ω over U $\Rightarrow$ we know what it is over X.

This is ~~trivial~~ in particular useful when $\omega = f^* \hat{\omega}$, f meromorphic map.