7 September 2026
MAT 4800

§3 (cont.)

Def: A path connected ~~set~~ topological space X is simply-connected if $\pi_1(X) = \{0\}$.

Thm 3.11 (HW): If X is simply connected, then every 2 curves in X which have the same endpoints are homotopic.

Proof: $u \circ v^{-1}$ ⇒ closed curve.

Example: open subset $U \subseteq X$ = manifold which is homeomorphic to a ball in $\mathbb{R}^n$.

Rmk: These sets are important in Čech Cohomology (later).

In AG, then the counterpart is affine open subsets.

Example 3.12 (i): is a way to use Van Kampen theorem.

3.15: Push forward:

Let $f: X \to Y$ be continuous, X & Y are path connected.

Choose a base point $a \in X$ & choose $b = f(a)$ the base point in Y.

Then we have a map:

$$f_* : \pi_1(X) \longrightarrow \pi_1(Y)$$

defined as follows:

⇒ On the level of curves:

If φ be a closed curve in X with endpoint a, then $f(\varphi)$ is closed curve in Y with endpoint b.

⇒ If φ_0 & φ_1 are homotopic in X then $f(\varphi_0)$ & $f(\varphi_1)$ are homotopic in Y.

So if an element of $\pi_1(X)$ is $[\varphi_0]$ then we defined:

$$f_*[\varphi_0] = [f(\varphi_0)] \in \pi_1(Y),$$

this is well-defined.

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§4. Branched & un-branched Covering

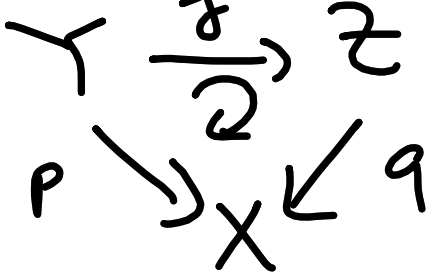
Def: (a) Let $p: Y \to X$ be continuous.

Fiber: $x \in X \Rightarrow$ fiber is $p^{-1}(x)$.

(1)

(2)

② Fiber preserving:



f is fiber preserving (for p & q): if $q \circ f = p$.

Explain: If we have: $p^{-1}(x) = f^{-1}(q^{-1}(x))$ (fiber over Y; fiber on Z)

③ Discrete map:
$\forall x \in X$, $f^{-1}(x)$ is discrete set in Y. □

Now these properties are relevant to holomorphic map of Riemann surfaces, by the following theorem:

Thm 4.2: Suppose X & Y are Riemann surfaces & $p: Y \to X$ is a non-constant holomorphic map. Then p is open & discrete.

Proof: p is open $\Rightarrow$ previously.
p discrete: if $y \in p^{-1}(x) \Rightarrow p(y) = x$,

So by the Identity principle

$\{y \in Y : p(y) = x\}$ is discrete. □

Def (Domain over a Riemann surface):

$p : Y \to X$ non-constant holomorphic map between Riemann surface $\Rightarrow Y$ is a domain over X. □

Def (Branch point): Same assumptions as before. Then $y_0 \in Y$ is a branch point for p if (also: ramification point) there is no open neighborhood U of y_0 so that $p|_U : U \to X$ is injective.

Rmk: y_0 is a branch point/ramification point for p (non-constant holomorphic) iff the local model of p near y_0 is $z \mapsto z^k$, where $k \geq 2$.

Lemma: y_0 is a branch point for p iff $p'(y_0) = 0$ (for all choices of coordinate charts near y_0).

Proof: Check this for the model map $z \to z^k$. (Here $y_0 = 0$)

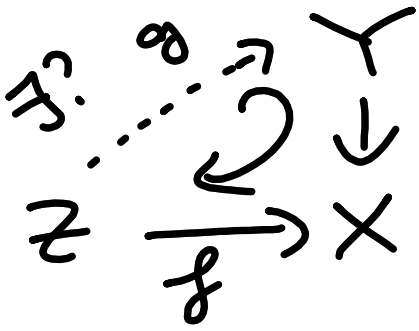
Example: $z \mapsto p(z) = e^z$.

Does it have any branch point?

Apply the lemma: $p'(z_0) = e^{z_0}$ never 0. So no branch point.

Rmk: No branch point is related to covering (later).

Def (Lifting problem of maps):



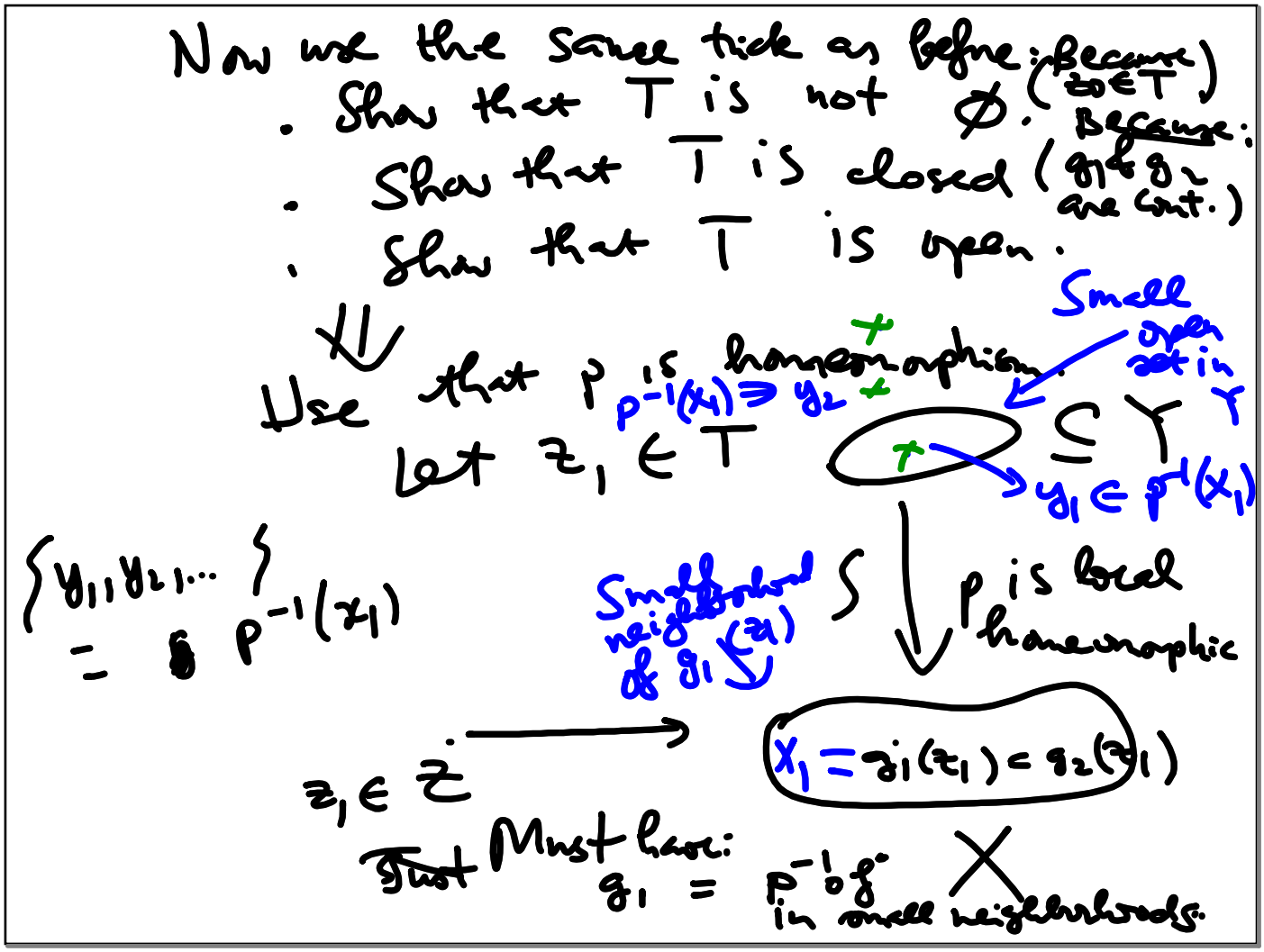
Question:
p, f continuous
Is there g continuous so that $p \circ g = f$?

Rmk: Lifting of mapping problem is used in many situations. Relevant to us is the holomorphic continuity (Riemann idea). Usually apply $Z = [0,1]$ (so we have lifting of curves).

3 important theorems about lifting of continuous maps:

Thm 4.8 (Uniqueness): We have at most one lifting (under some reasonable conditions). Here the reasonable conditions are: ① X & Y are Hausdorff spaces ② $p: Y \to X$ is local homeomorphism (important special case: covering map) ③ Z connected.

Proof: Let g_1, g_2 to be 2 liftings, with $g_1(z_0) = g_2(z_0)$. Want to show $g_1 = g_2$. Then suffices to show that the set $T = \{z \in Z : g_1(z) = g_2(z)\}$ is equal to Z.


Now use the same trick as before: (Because $\tilde{z}_0 \in T$)
. Show that T is not $\emptyset$.
. Show that $\overline{T}$ is closed (Because: g_1, g_2 are cont.)
. Show that $\overline{T}$ is open.
$\Downarrow$
Use that p is homeomorphism.
Let $z_1 \in T$
$p^{-1}(x_1) \ni y_2$
$y_1 \in p^{-1}(x_1)$
Small open set in Y
$\subseteq Y$
$\{y_1, y_2, \dots\} = p^{-1}(x_1)$
Small neighborhood of $g_1(z_1)$
p is local homeomorphic
$z_1 \in \tilde{z}$
$x_1 = g_1(z_1) = g_2(z_1)$
Must have: $g_1 = p^{-1} \circ g$ in small neighborhoods.

Similar for g_2. $\square$