In particular, in the space hol(Ω) the C^∞ convergence coincides with the L^1_{loc} convergence, indeed with any "weak" (except from pointwise) convergence. But the main consequence is about the so-called "normal families" of holomorphic functions.

(also Montel's)

THEOREM 1.2.11 (Stjelties-Vitali). *Let $\{f_\nu\}$ be a bounded sequence in* hol(Ω). *Then, there is a subsequence $\{f_{\nu_k}\}$ uniformly convergent on compact subsets of Ω to a limit $f \in$* hol(Ω).

PROOF. By Theorem 1.2.10, "$\{f_\nu\}$ bounded" implies "$\{\partial_{z_j} f_\nu\}$ bounded" on compact subsets. Hence $\{f_\nu\}$ is equicontinuous and by Arzela's theorem it converges to a limit. The limit is also holomorphic, just by interchanging $\partial_{\bar z_j}$ with "lim". □

1.3. Analytic functions and power series

The domain of convergence of a power series of one complex variable is a disc. For a power series in several complex variables it shows a more interesting structure. It also serves as the first non-trivial example of a "forced" extension for holomorphic functions which is not possessed by functions of one variable. Let

$$\sum_{|\alpha|=0}^{+\infty} a_\alpha z^\alpha \tag{1.3.1}$$

be a power series centered at 0.

DEFINITION 1.3.1. We denote by D the domain of convergence of the power series (1.3.1) which consists of the set of points in whose neighborhood the power series converges normally, that is, absolutely uniformly.

DEFINITION 1.3.2. We denote by B the set of "boundedness" of (1.3.1), that is, the set of points z such that $|a_\alpha z^\alpha| < c$ for any α.

The two above sets are related by the following theorem, which is an elementary application of Abel's theorem.

THEOREM 1.3.3. *We have $D = \overset{\circ}{B}$.*

PROOF. D is obviously open and contained in $\overset{\circ}{B}$. Conversely, if $z_o \in \overset{\circ}{B}$, then there are a point $w \in B$, positive numbers $k_j < 1$ and a neighborhood V of z_o such that $|z_j| \le k_j |w_j|$ for any j and $z \in V$.

Let $k = (k_1, ..., k_n)$; it follows that

$$\begin{aligned}\sum_{\alpha} |a_\alpha z^\alpha| &\leq c \sum_{\alpha} k^\alpha \\ &= c \prod_{j=1,...,n} (1-k_j)^{-1} \quad \text{for any } z \in V.\end{aligned} \tag{1.3.2}$$

□

Therefore, the domain of convergence D of a power series shows its first two interesting features:

- It is invariant under rotations: if $z \in D$, then $e^{i\theta} z \in D$ for any $\theta \in [0, 2\pi]^n$.
- If $w \in D$, then, for any z which satisfies $|z_j| \leq |w_j|$ for all j, we also have $z \in D$.

An open set which enjoys the first property is called a "Reinhardt domain": it can be described through its representative in the space of moduli $|D| \subset (\mathbb{R}^+)^n$ defined by $|D| := \{(|z_1|, ..., |z_n|) : z \in D\}$. A Reinhardt domain which enjoys the second property is called "complete": it is therefore characterized as a union of polydiscs centered at 0. Therefore, for a domain of $\mathbb{C}^n$ to be the domain of convergence of a power series, we have to require it to be Reinhardt and complete. But not all complete Reinhardt domains are precisely the convergence domain of a power series. Some additional geometric requirements must be fulfilled.

THEOREM 1.3.4. *D is "logarithmically convex" in the sense that the set $D^* := \{\xi \in \mathbb{R}^n : \ \xi_j = \log|z_j|, \ j = 1, ..., n, \ for\ z \in D\}$ is convex.*

PROOF. Remember that $D = \overset{\circ}{B}$; hence, what we have to prove is that

$$\begin{cases} |a_\alpha z^\alpha| < c, \\ |a_\alpha w^\alpha| < c \end{cases} \tag{1.3.3}$$

implies for $\lambda \in [0, 1]$

$$|a_\alpha z^{\lambda\alpha} w^{(1-\lambda)\alpha}| < c. \tag{1.3.4}$$

But this is a straightforward calculation.

□

We compare now analytic functions and convergent power series. We have already seen that they coincide over polydiscs, hence over the union of those which are centered at the same point: this follows easily from analytic continuation. In particular, over a Reinhardt complete (non-empty) domain, a holomorphic function is always represented as a sum of a power series centered at 0. (Of course for a general domain

$\Omega \subset \mathbb{C}^n$, a holomorphic function is a family of convergent power series but not a single one, in general.) We see now that a Reinhardt domain need not be complete for this conclusion to be true.

THEOREM 1.3.5. *Let Ω be a connected Reinhardt domain which contains* 0 *and let $f \in \mathrm{hol}(\Omega)$; then we have*

$$f(z) = \sum_{\alpha} \frac{f^{(\alpha)}(0)}{\alpha!} z^{\alpha},$$

with normal convergence in Ω.

PROOF. We denote by Ω_ϵ the connected component of 0 in the set $\{z \in \Omega : d(z, \mathbb{C}^n \setminus \Omega) > \epsilon|z|\}$. The sets Ω_ϵ are increasing for $\epsilon \searrow 0$ and their union over ϵ covers the whole of Ω since this is connected. We denote by $P_{1+\epsilon}$ the polydisc $P_{1+\epsilon} = \{t \in \mathbb{C}^n : \ |t_j| < 1 + \epsilon \ \text{ for any } j = 1, ..., n\}$ and define

$$g(z) = (2\pi i)^{-n} \int_{\partial_0 P_{1+\epsilon}} \frac{f(t_1 z_1, ..., t_n z_n)}{(t_1 - 1)...(t_n - 1)} dt_1 \wedge ... \wedge dt_n. \tag{1.3.5}$$

Since $z \in \Omega_\epsilon$, then $(1+\epsilon)z \in \Omega$; since $t \in \partial_0 P_{1+\epsilon}$ and Ω is Reinhardt, then $tz \in \Omega$. Thus, the integral is defined and produces a smooth function of z. By interchanging $\partial_{\bar{z}_j}$ with integration, we see that g is holomorphic. When t leaves $\partial_0 P_{1+\epsilon}$ and enters into $P_{1+\epsilon}$, then tz is no longer contained in Ω, in general. But this is true for small z since Ω contains a neighborhood of 0. For these values of z, the substitutions $\zeta := tz$ and $P^z_{1+\epsilon} = \{tz : t \in P_{1+\epsilon}\}$ inside (1.3.5) yield $g(z) = (2\pi i)^{-n} \int_{\partial_0 P^z_{1+\epsilon}} \frac{f(\zeta)}{(\zeta_1 - z_1)...(\zeta_n - z_n)} d\zeta_1 \wedge ... \wedge d\zeta_n$. But the term in the right side is the Cauchy integral of f at z for the polydisc $P^z_{1+\epsilon}$ which is completely contained in the region where f is holomorphic. Hence $f(z) = g(z)$ for those values of z, but then in fact $f \equiv g$ on Ω_ϵ by analytic continuation. We turn our attention now to the power series

$$\frac{1}{(t_1 - 1)...(t_n - 1)} = \sum_{\alpha} t^{-\alpha - 1},$$

$+\varepsilon$

with normal convergence for $t \in \partial_0 P_{1-\epsilon}$. If we take $\sum_\alpha$ out of the integral on the right side of (1.3.5), we get $f = \sum_\alpha f_\alpha$ where the f_α are the analytic functions defined by

$$f_\alpha(z) = (2\pi i)^{-n} \int_{\partial_0 P_{1+\epsilon}} \frac{f(tz)}{t^{\alpha+1}} dt_1 \wedge ... \wedge dt_n.$$

In the same way as for the function g previously discussed, we can see that $f_\alpha(z) = \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha$ on Ω_ϵ. But then f is a sum of a power series. $\square$

Since a holomorphic function on a Reinhardt domain containing 0 is represented by a power series, then the function extends analytically wherever the power series converges. Now, the first points to be added to the initial domain are those of its "completion", the points z such that for some $w \in \Omega$ we have $|z_j| \leq |w_j|$ for any j and then also the points of its "logarithmically convex hull", the points z for which there are w^1 and w^2 in Ω such that $\log|z_j| = \lambda \log|w_j^1| + (1-\lambda)\log|w_j^2|$ for any $j \leq n$ and for $\lambda \in [0,1]$. All these points were unnaturally "missing" from the initial domain of f. These remarks are collected in the following statement.

THEOREM 1.3.6. *Let Ω be a connected Reinhardt domain containing 0 and let $\tilde{\Omega}$ be the smallest complete logarithmically convex Reinhardt domain containing Ω. Then we have an extension mapping*

$$\mathrm{hol}(\Omega) \overset{\mathrm{ext}}{\dashrightarrow} \mathrm{hol}(\tilde{\Omega}).$$

REMARK 1.3.7. We have purposely included the pleonastic word "complete" in the above statement: but a Reinhardt domain logarithmically convex and containing 0 is always complete. In fact, let $w \in \Omega$: we know that $\epsilon^\lambda |w|^{1-\lambda} := (\epsilon_1^\lambda |w_1|^{1-\lambda}, \dots, \epsilon_n^\lambda |w_n|^{1-\lambda})$ must belong to $|\Omega|$ for any real small ϵ and for any $\lambda \in [0,1]$. But the set of these points whose image in the space of moduli is $\epsilon^\lambda |w|^{1-\lambda}$ covers the whole polydisc $\{z : \ |z_j| \leq |w_j| \ \text{ for any } j\}$.

We conclude this discussion by remarking that, conversely, $\tilde{\Omega}$ is the maximal set of analytic extension or, in other words, there is a holomorphic function which does not extend beyond $\tilde{\Omega}$. This is a consequence of Theorem 1.8.8 whose proof can be found in detail for instance in [**49**, Corollary 2.5.8]. On the other hand this function is indeed represented by a unique power series. Thus, a Reinhardt, complete, logarithmically convex domain is the precise domain of convergence of a power series.

1.4. Subharmonic functions

It is clear from the end of the last section that our most attractive task is now to describe the natural "domains" of holomorphic functions. For this purpose we need to develop some preliminary technology; it will also be the main tool in solving the problem of separate analyticity. Let us declare from the very beginning that this is strictly a one-variable theory. We recall that a C^2 function h on a domain Ω of $\mathbb{C}$ with coordinate $z = x + iy$ is said to be harmonic when $\partial_z \partial_{\bar{z}} h = 0$.

DEFINITION 1.4.1. A real function φ on $\Omega \subset \mathbb{C}$ with values in $[-\infty, +\infty)$ is subharmonic when