However, we can already at this point discuss equivalent forms of the condition just obtained.

Definition 2.6.6. *If K is a compact subset of the open set $\Omega \subset \mathbf{C}^n$ we define the $P(\Omega)$-hull $\hat{K}_\Omega^P$ of K by*

$$\hat{K}_\Omega^P = \{z; z \in \Omega, u(z) \leq \sup_K u \text{ for all } u \in P(\Omega)\}.$$

It is clear that the $P(\Omega)$-hull of K is contained in the $A(\Omega)$-hull of K.

Theorem 2.6.7. *If Ω is an open set in $\mathbf{C}^n$, the following conditions are equivalent:*

(i) $-\log \delta(z, \complement\Omega)$ *is plurisubharmonic in Ω if δ is defined as in Theorem 2.5.4.*

(ii) *There exists a continuous plurisubharmonic function u in Ω such that*

$$\Omega_c = \{z; z \in \Omega, u(z) < c\} \subset\subset \Omega$$

for every $c \in \mathbf{R}$.

(iii) $\hat{K}_\Omega^P \subset\subset \Omega$ *if* $K \subset\subset \Omega$.

Proof. If (i) is fulfilled, we only have to set $u(z) = |z|^2 - \log \delta(z, \complement\Omega)$ to get a function satisfying (ii). That (ii) implies (iii) is obvious, so we need only prove that (iii) implies (i). Let $z_0 \in \Omega$, $0 \neq w \in \mathbf{C}^n$, and choose $r > 0$ so that

$$D = \{z_0 + \tau w; |\tau| \leq r\} \subset \Omega.$$

Let $f(\tau)$ be an analytic polynomial such that

$$-\log \delta(z_0 + \tau w, \complement\Omega) \leq \operatorname{Re} f(\tau), \qquad |\tau| = r,$$

that is,

$$\delta(z_0 + \tau w, \complement\Omega) \geq |e^{-f(\tau)}|, \qquad |\tau| = r. \tag{2.6.2}$$

We want to prove the same inequality when $|\tau| \leq r$. To do so, we take any vector $a \in \mathbf{C}^n$ with $\delta(a) < 1$ and consider for $0 \leq \lambda \leq 1$ the mapping

$$\tau \to z_0 + \tau w + \lambda a e^{-f(\tau)}, \qquad |\tau| \leq r.$$

We denote its range by D_λ; clearly $D_0 = D$. Put

$$\Lambda = \{\lambda; 0 \leq \lambda \leq 1, D_\lambda \subset \Omega\}.$$

It is obvious that Λ is an open subset of $[0,1]$ and to prove that Λ is equal to the whole interval we shall prove that Λ is closed. Let K be the compact set

$$K = \{z_0 + \tau w + \lambda a e^{-f(\tau)}, |\tau| = r, 0 \le \lambda \le 1\},$$

which is contained in Ω by (2.6.2). If $u \in P(\Omega)$ and $\lambda \in \Lambda$, then

$$\tau \to u(z_0 + \tau w + \lambda a e^{-f(\tau)})$$

is subharmonic in a neighborhood of the disc $|\tau| \le r$, which proves that

$$u(z_0 + \tau w + \lambda a e^{-f(\tau)}) \le \sup_K u \quad \text{if} \quad |\tau| \le r.$$

Hence $D_\lambda \subset \hat{K}_\Omega{}^P$ for every $\lambda \in \Lambda$, and this implies that Λ is closed, for $\hat{K}_\Omega{}^P$ is relatively compact in Ω by (iii). Thus $D_1 \subset \Omega$, that is,

$$z_0 + \tau w + a e^{-f(\tau)} \in \Omega \quad \text{if } \delta(a) < 1 \text{ and } |\tau| \le r,$$

so that $|\delta(z_0 + \tau w, \complement\,\Omega)| \ge |e^{-f(\tau)}|$ if $|\tau| \le r$, or

$$-\log \delta(z_0 + \tau w, \complement\,\Omega) \le \operatorname{Re} f(\tau), \qquad |\tau| \le r.$$

This proves (i).

Definition 2.6.8. *The open set $\Omega \subset \mathbf{C}^n$ is called pseudoconvex if the equivalent conditions in Theorem 2.6.7 are fulfilled.*

Since the supremum of a family of plurisubharmonic functions is plurisubharmonic if it is continuous, we obtain from condition (i) in Theorem 2.6.7:

Theorem 2.6.9. *If Ω_α is a pseudoconvex open set for every α in an index set A, then the interior Ω of $\cap_{\alpha \in A} \Omega_\alpha$ is also pseudoconvex.*

The corresponding statement for holomorphy domains was given in Corollary 2.5.7. However, the next theorem is by no means obvious for domains of holomorphy and is, in fact, in that case essentially equivalent to the identity of holomorphy domains and pseudoconvex domains which we shall prove in Chapter IV.

Theorem 2.6.10. *Let Ω be an open set in $\mathbf{C}^n$. If to every point in $\bar{\Omega}$ there is a neighborhood ω such that $\omega \cap \Omega$ is pseudoconvex, then Ω is pseudoconvex.*

The condition in the theorem is of course only a restriction on $\partial\Omega$. Loosely stated, the theorem means that pseudoconvexity is a local property of the boundary.

Proof of Theorem 2.6.10. Let $z_0 \in \partial\Omega$ and choose a neighborhood ω of z_0 according to the hypothesis. Then $\delta(z, \complement\Omega) = \delta(z, \complement(\Omega \cap \omega))$ for all z sufficiently close to z_0. Hence $-\log \delta(z, \complement\Omega)$ is plurisubharmonic in a neighborhood of every point on $\partial\Omega$, so there is a closed set $F \subset \Omega$ such that $-\log \delta(z, \complement\Omega)$ is plurisubharmonic in $\Omega \setminus F$. Now take a continuous function $\varphi \in P(\mathbf{C}^n)$ (for example, a convex increasing function of $|z|^2$) such that $\varphi(z) > -\log \delta(z, \complement\Omega)$ when $z \in F$, and $\varphi(z) \to \infty$ when $|z| \to \infty$. Then $u(z) = \sup(\varphi(z), -\log \delta(z, \complement\Omega))$ is in $P(\Omega)$, for $u = \varphi$ in a neighborhood of F and the supremum of two plurisubharmonic functions is plurisubharmonic. It is clear that u satisfies condition (ii) in Theorem 2.6.7, which proves that Ω is pseudoconvex.

For reference in Chapter IV we give a property of the $P(\Omega)$-hull which seems much stronger than that in Definition 2.6.6.

Theorem 2.6.11. *Let Ω be a pseudoconvex open set in $\mathbf{C}^n$, let K be a compact subset of Ω, and ω an open neighborhood of $\hat{K}_\Omega^P$. Then there exists a function $u \in C^\infty(\Omega)$ such that*

(a) *u is strictly plurisubharmonic, that is, the hermitian form in* (2.6.1) *is strictly positive definite for every $z \in \Omega$.*
(b) *$u < 0$ in K but $u > 0$ in $\Omega \cap \complement\omega$*
(c) *$\{z; z \in \Omega, u(z) < c\} \subset\subset \Omega$ for every $c \in \mathbf{R}$.*

Proof. We shall first construct a continuous function $v \in P(\Omega)$ satisfying (b) and (c). To do so we choose a function u_0 with the properties listed in (*ii*) of Theorem 2.6.7 Adding a constant to u_0, if necessary, we may assume that $u_0 < 0$ in K Set

$$K' = \{z; z \in \Omega, u_0(z) \le 2\}, L = \{z; z \in \Omega \cap \complement\omega, u_0(z) \le 0\}.$$

These sets are both compact. For every $z \in L$ we can choose a function $w \in P(\Omega)$ such that $w(z) > 0$ but $w < 0$ in K. By means of a regularization as in Theorem 2.6.3 we obtain a continuous plurisubharmonic function w_1 in a neighborhood of K', with $w_1 < 0$ in K and $w_1 > 0$ in a neighborhood of z Since L is compact, we can now use the Borel–Lebesgue lemma and the fact that the supremum of a finite family of plurisubharmonic functions is plurisubharmonic to construct a continuous plurisubharmonic function w_2 in a neighborhood of K' such that $w_2 > 0$ in a neighborhood of L and $w_2 < 0$ in K. Let C be the maximum of w_2 in K' and set for $z \in \Omega$,

$$v(z) = \sup(w_2(z), Cu_0(z)) \quad \text{if } u_0(z) < 2, \qquad v(z) = Cu_0(z) \quad \text{if } u_0(z) > 1.$$

The two definitions agree when $1 < u_0(z) < 2$, so v is a continuous plurisubharmonic function in Ω, which obviously satisfies (b) and (c).

Let

$$\Omega_c = \{z\,;\, z \in \Omega,\, v(z) < c\}.$$

If with the notations of Theorem 2.6.3 we set

$$v_j(z) = \int_{\Omega_{j+1}} v(\zeta)\varphi((z-\zeta)/\varepsilon)\varepsilon^{-2n}\, d\lambda(\zeta) + \varepsilon|z|^2 \quad (j = 0,1,\cdots)$$

and ε is chosen sufficiently small (depending on j), we obtain a function $v_j \in C^\infty(\mathbf{C}^n)$ which is $> v$ and strictly plurisubharmonic in a neighborhood of $\bar{\Omega}_j$. By suitable choice of ε we can also achieve that $v_0 < 0$ and $v_1 < 0$ in K and $v_j < v + 1$ in Ω_j for $j = 1, 2, \cdots$. Now take a convex function $\chi \in C^\infty(\mathbf{R})$ such that $\chi(t) = 0$ when $t < 0$ and $\chi'(t) > 0$ when $t > 0$. Then $\chi(v_j + 1 - j)$ is strictly plurisubharmonic in a neighborhood of $\bar{\Omega}_j \setminus \Omega_{j-1}$. We can therefore choose successively positive numbers $a_1, a_2, \cdots$ so that

$$u_m = v_0 + \sum_1^m a_j \chi(v_j + 1 - j)$$

is $> v$ and strictly plurisubharmonic in a neighborhood of $\bar{\Omega}_m$. We have $u_m = u_l$ in Ω_j if l and m are $> j$, so $u = \lim u_m$ exists and is a strictly plurisubharmonic C^∞ function in Ω. Since $u = v_0 < 0$ in K and $u > v$ in Ω, the properties (a), (b), (c) follow.

If Theorem 2.6.11 is applied to $\omega = \Omega \setminus \{x\}$, where $x \notin \hat{K}_\Omega{}^P$, it follows that we can restrict u to $P(\Omega) \cap C^\infty(\Omega)$ in Definition 2.6.6. Hence $\hat{K}_\Omega{}^P$ is closed and therefore compact if K is a compact subset of Ω and Ω is pseudoconvex.

We shall now examine when an open set with a C^2 boundary is pseudoconvex.

Theorem 2.6.12. *Let $\Omega \subset \mathbf{C}^n$ be an open set with a C^2 boundary; let $\Omega = \{z\,;\, \rho(z) < 0\}$ where ρ is in C^2 in a neighborhood of $\bar{\Omega}$ and* $\operatorname{grad} \rho \neq 0$ *on $\partial\Omega$ Then Ω is pseudoconvex if and only if*

$$(2.6.3) \quad \sum_{j,k=1}^n \partial^2\rho/\partial z_j \partial \bar{z}_k\, w_j \bar{w}_k \geq 0 \quad \text{when } z \in \partial\Omega \text{ and} \sum_1^n \partial\rho/\partial z_j\, w_j = 0.$$

Condition (2.6.3) is called the *Levi condition*; $\partial\Omega$ is also said to be *pseudoconvex* if (2.6.3) holds.

Proof. If ρ_1 is another function satisfying the hypotheses in the theorem, then $\rho_1 = h\rho$ with $h > 0$ in a neighborhood of $\bar{\Omega}$. Hence

$$\sum_{j,k=1}^n \partial^2\rho_1/\partial z_j \partial \bar{z}_k\, w_j \bar{w}_k = h \sum_{j,k=1}^n \partial^2\rho/\partial z_j \partial \bar{z}_k\, w_j \bar{w}_k \quad \text{if } \rho = \sum_1^n \partial\rho/\partial z_j\, w_j = 0,$$

which proves that (2.6.3) is independent of the choice of ρ. To prove that there is a function satisfying (2.6.3) if Ω is pseudoconvex, we set

$$\rho(z) = -\delta(z, \complement\Omega) \quad \text{in } \Omega, \qquad \rho(z) = \delta(z, \Omega) \quad \text{in } \complement\Omega$$

where δ is, for example, the Euclidean metric. Then $\rho \in C^2$ near the boundary of Ω. (This follows from the implicit function theorem.) At points in Ω sufficiently close to $\partial\Omega$, the plurisubharmonicity of $-\log\delta$ means that

$$\sum_{j,k=1}^{n} (-\delta^{-1}\, \partial^2\delta/\partial z_j\partial\bar{z}_k + \delta^{-2}\, \partial\delta/\partial z_j \partial\delta/\partial\bar{z}_k)\, w_j\bar{w}_k \geq 0.$$

Hence

$$\sum_{j,k=1}^{n} \partial^2\rho/\partial z_j\partial\bar{z}_k\, w_j\bar{w}_k \geq 0 \quad \text{if } \sum_{1}^{n} \partial\rho/\partial z_j\, w_j = 0.$$

A passage to the limit shows that this is also true on $\partial\Omega$.

In proving the converse we may by the first part of the proof assume that (2.6.3) is satisfied with ρ defined as above. Assume now that with $w \in \mathbf{C}^n$

$$c = \frac{\partial^2}{\partial\tau\partial\bar{\tau}} \log\delta(z + \tau w, \complement\Omega) > 0 \quad \text{when } \tau = 0,$$

for some z so close to $\partial\Omega$ that $\delta \in C^2$ at z. Then we have by Taylor's formula

$$\log\delta(z + \tau w, \complement\Omega) = \log\delta(z, \complement\Omega) + \operatorname{Re}(A\tau + B\tau^2) + c|\tau|^2 + o(|\tau|^2), \quad \tau \to 0.$$

Here A and B are constants. Now choose $a \in \mathbf{C}^n$ with $\delta(a) = \delta(z, \complement\Omega)$ so that $z + a \in \partial\Omega$, and set

$$z(\tau) = z + \tau w + a \exp(A\tau + B\tau^2).$$

Then we have

$$\delta(z(\tau), \complement\Omega) \geq \delta(z + \tau w, \complement\Omega) - \delta(a)|\exp(A\tau + B\tau^2)|$$
$$\geq \delta(a)(e^{c|\tau|^2/2} - 1)|e^{A\tau + B\tau^2}|$$

when $|\tau|$ is sufficiently small. Since $\delta(z(0), \complement\Omega) = 0$, we conclude that $(\partial/\partial\tau)\delta(z(\tau), \complement\Omega) = 0$ and that $(\partial^2/\partial\tau\partial\bar{\tau})\delta(z(\tau), \complement\Omega) > 0$ when $\tau = 0$. With the notation ρ used above, this means, since $z(\tau)$ is an analytic function of τ, that

$$\sum_{1}^{n} \partial\rho/\partial z_j\, z_j'(0) = 0, \qquad \sum_{j,k=1}^{n} \partial^2\rho/\partial z_j\partial\bar{z}_k\, z_j'(0)\,\overline{z_k'(0)} < 0.$$

This contradicts (2.6.3) at $z(0)$, which completes the proof. (Note the analogy of this proof with that of Theorem 2.6.7.)

The necessity of (2.6.3) is also easily proved directly. In fact, when (2.6.3) is violated, one can even prove a local extension theorem analogous to Theorem 2.3.2'.

Theorem 2.6.13. *Let ρ be a C^4 function in a neighborhood ω of z_0 such that $\rho(z_0) = 0$ but* grad $\rho(z_0) \neq 0$. *Further, assume that*

(2.6.4)

$$\sum_{j,k=1}^{n} \partial^2\rho(z_0)/\partial z_j\partial\bar{z}_k\, w_j\bar{w}_k < 0 \quad \textit{for some } w \in \mathbf{C}^n \textit{ with } \sum_1^n w_j\partial\rho(z_0)/\partial z_j = 0.$$

Then there exists a neighborhood $\omega' \subset \omega$ of z_0 such that, for every $u \in C^4(\omega')$ satisfying the tangential Cauchy–Riemann equations, $\bar\partial u \wedge \bar\partial\rho = 0$ on $\{z; z \in \omega', \rho(z) = 0\}$, one can find $U \in C^1(\omega')$ so that $U = u$ when $\rho = 0$, and $\bar\partial U = 0$ in $\omega_+' = \{z; z \in \omega', \rho(z) \geq 0\}$.

Proof. After a linear change of coordinates we may assume that $z_0 = 0$ and that

$$\rho(x) = x_{2n} + A(x) + O(|x|^3)$$

where A is a quadratic form. Taylor's formula gives

$$A(x) = \sum_{j,k=1}^{n} z_j\bar{z}_k\, \partial^2\rho(0)/\partial z_j\partial\bar{z}_k + \operatorname{Re} \sum_{j,k=1}^{n} z_j z_k\, \partial^2\rho(0)/\partial z_j\partial z_k.$$

If we make the analytic change of variables

$$z_j' = z_j \quad \text{if } j \leq n-1,\ z_n' = z_n + i \sum_{j,k=1}^{n} z_j z_k\, \partial^2\rho(0)/\partial z_j\partial z_k,$$

the Taylor expansion of ρ assumes the simpler form

$$\rho = \operatorname{Im} z_n' + \sum_{j,k=1}^{n} z_j'\overline{z_k'}\, \partial^2\rho(0)/\partial z_j\partial\bar{z}_k + O(|z'|^3).$$

To simplify notations we may therefore without restriction assume that the original coordinates were chosen so that

$$\rho = \operatorname{Im} z_n + \sum_{j,k=1}^{n} A_{jk} z_j\bar{z}_k + O(|z|^3),$$

where (A_{jk}) is a hermitian symmetric matrix. The hypothesis (2.6.4) means