3 Convexity

3.1 Many Notions of Convexity

The concept of convexity goes back to the work of Archimedes, who used the idea in his axiomatic treatment of arc length. The notion was treated sporadically, and in an ancillary fashion, by Fermat, Cauchy, Minkowski, and others. It was not until the 1930s, however, that the first treatise on convexity (by Bonneson and Fenchel [1]) appeared. An authoritative discussion of the history of convexity can be found in Fenchel [1].

One of the most prevalent and classical definitions of convexity is as follows: A subset $S \subseteq \mathbb{R}^N$ is said to be convex if whenever $P, Q \in S$ and $0 \leq \lambda \leq 1$, then $(1-\lambda)P + \lambda Q \in S$. In the remainder of this book we shall refer to a set or domain satisfying this condition as *geometrically convex*. From the point of view of analysis, this definition is of little use. We say this because the definition is *nonquantitative, nonlocal*, and *not formulated in the language of functions*. Put slightly differently, we have known since the time of Riemann that the most useful conditions in geometry are differential conditions. Thus we wish to find a differential characterization of convexity. We begin this chapter by relating classical notions of convexity to more analytic notions. All these ideas are properly a part of *real analysis*, so we restrict attention to $\mathbb{R}^N$.

Let $\Omega \subseteq \mathbb{R}^N$ be a domain with C^1 boundary. Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a C^1 *defining function* for Ω. Recall (Section 0.1 and Exercise 7 at the end of Chapter 1) that such a function has these properties:

1. $\Omega = \{x \in \mathbb{R}^N : \rho(x) < 0\}$
2. ${}^c\bar{\Omega} = \{x \in \mathbb{R}^N : \rho(x) > 0\}$
3. $\operatorname{grad} \rho(x) \neq 0 \quad \forall x \in \partial\Omega$

If $k \geq 2$ and the boundary of Ω is a C^k manifold in the usual sense (see Appendix I), then it is straightforward to manufacture a C^1 (indeed a C^k) defining function for Ω by using the signed distance-to-the-boundary function.

DEFINITION 3.1.1 Let $\Omega \subseteq \mathbb{R}^N$ have C^1 boundary and let ρ be a C^1 defining function. Let $P \in \partial\Omega$. An N-tuple $w = (w_1, \ldots, w_N)$ of real numbers is called a *tangent vector* to $\partial\Omega$ at P if

$$\sum_{j=1}^{N} (\partial\rho/\partial x_j)(P) \cdot w_j = 0.$$

We write $w \in T_P(\partial\Omega)$.

Of course this definition makes sense only if it is independent of the choice of ρ. We shall address that issue in a moment. First, if P is a boundary point of a domain Ω with C^1 boundary, then we let ν_P denote the unit outward normal to $\partial\Omega$ at P. It should be observed that the condition defining tangent vectors simply mandates that $w \perp \nu_P$ at P. And, after all, we know from calculus that $\nabla\rho$ is the normal ν_P and that the normal is uniquely determined and independent of the choice of ρ. In principle, this settles the well-definedness issue.

However this point is so important and the point of view that we are considering is so pervasive that further discussion is warranted. The issue is this: if $\hat{\rho}$ is another defining function for Ω, then it should give the same tangent vectors as ρ at any point $P \in \partial\Omega$. The key to seeing that this is so is to write $\hat{\rho}(x) = h(x) \cdot \rho(x)$, for h a function that is nonvanishing near $\partial\Omega$. Then, for $P \in \partial\Omega$,

$$\begin{aligned} \sum_{j=1}^{N} (\partial\hat{\rho}/\partial x_j)(P) \cdot w_j &= h(P) \cdot \left(\sum_{j=1}^{N} (\partial\rho/\partial x_j)(P) \cdot w_j \right) \\ &\quad + \rho(P) \cdot \left(\sum_{j=1}^{N} (\partial h/\partial x_j)(P) \cdot w_j \right) \\ &= h(P) \cdot \left(\sum_{j=1}^{N} (\partial\rho/\partial x_j)(P) \cdot w_j \right) \\ &\quad +0, \end{aligned} \tag{3.1.2}$$

because $\rho(P) = 0$. Thus w is a tangent vector at P vis-à-vis ρ if and only if w is a tangent vector vis-à-vis $\hat{\rho}$. But why does h exist?

After a change of coordinates, it is enough to assume that we are dealing with a piece of $\partial\Omega$ that is a piece of flat, $(N-1)$-dimensional real hypersurface (just use the implicit function theorem). Thus we may take $\rho(x) = x_N$ and

$P = 0$. Then any other defining function $\hat{\rho}$ for $\partial\Omega$ near P must have the Taylor expansion

$$\hat{\rho}(x) = c \cdot x_N + \mathcal{R}(x)$$

about 0. Here $\mathcal{R}$ is a remainder term satisfying $\mathcal{R}(x) = o(|x|)$. (There is no loss of generality to take $c = 1$, and we do so in what follows.) Thus we wish to define

$$h(x) = \frac{\hat{\rho}(x)}{\rho(x)} = 1 + \mathcal{S}(x).$$

Here $\mathcal{S}(x) \equiv \mathcal{R}(x)/x_N$ and $\mathcal{S}(x) = o(1)$ as $x_N \to 0$. Since this remainder term involves a derivative of $\hat{\rho}$, it is plain that h is not even differentiable. (An explicit counterexample is given by $\hat{\rho}(x) = x_N \cdot (1 + |x_N|)$.) Thus the program that we attempted in equation (3.1.2) is apparently flawed.

However an inspection of the explicit form of the remainder term $\mathcal{R}$ reveals that because $\hat{\rho}$ is constant on $\partial\Omega$, h as defined above *is* continuously differentiable *in tangential directions.* That is, for tangent vectors w (vectors that are orthogonal to ν_P), the derivative

$$\sum_j \frac{\partial h}{\partial x_j}(P) w_j$$

is defined. Thus it does indeed turn out that our definition of tangent vector is well posed when it is applied to vectors *that are already known to be tangent vectors* by the geometric definition $w \cdot \nu_P = 0$. For vectors that are *not* geometric tangent vectors, an even simpler argument shows that

$$\sum_j \frac{\partial \hat{\rho}}{\partial x_j}(P) w_j \neq 0$$

if and only if

$$\sum_j \frac{\partial \rho}{\partial x_j}(P) w_j \neq 0.$$

Thus Definition 1.1 is well posed. Questions similar to the one just discussed will come up later when we define convexity using C^2 defining functions (and also when we define the concept of pseudoconvexity). They are resolved in just the same way and we shall leave details to the reader.

The reader should check that the discussion above proves the following: If $\rho, \tilde{\rho}$ are C^k defining functions for a domain Ω, then there is a C^{k-1} function h defined near $\partial\Omega$ such that $\rho = h \cdot \tilde{\rho}$.

This somewhat protracted discussion of a small technical point seems necessary because it is recorded incorrectly in most places in the literature (including the first edition of this book).

3.1.1 The Analytic Definition of Convexity

For convenience, we restrict attention for this subsection to *bounded* domains. Many of our definitions would need to be modified and extra arguments would need to be given in proofs were we to consider unbounded domains as well.

DEFINITION 3.1.3 Let $\Omega \subset\subset \mathbb{R}^N$ be a domain with C^2 boundary and ρ a defining function for Ω. Fix a point $P \in \partial\Omega$. We say that $\partial\Omega$ is (weakly) *convex* at P if

$$\sum_{j,k=1}^{N} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k \geq 0, \qquad \forall w \in T_P(\partial\Omega).$$

We say that $\partial\Omega$ is *strongly convex* at P if the inequality is strict whenever $w \neq 0$.

If $\partial\Omega$ is convex (resp. strongly convex) at each boundary point, then we say that Ω is convex (resp. strongly convex).

The quadratic form

$$\left(\frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) \right)_{j,k=1}^{N}$$

is frequently called the "real Hessian" of the function ρ. This form carries considerable geometric information about the boundary of Ω. It is, of course, closely related to the second fundamental form of Riemannian geometry (see B. O'Neill [1]).

There is a technical difference between strong and strict convexity that we shall not discuss here (see L. Lempert [2] for details). It is common to use either of the words strong or strict to mean that the inequality in the last definition is strict when $w \neq 0$. The reader may wish to verify that at a strongly convex boundary point, all curvatures are positive (in fact, one may, by the positive definiteness of the matrix $\left(\partial^2 \rho / \partial x_j \partial x_k\right)$, impose a change of coordinates at P so that the boundary of Ω agrees with a ball up to and including second order at P). It is also the case that any strongly convex boundary point P is *extreme:* If $x, y \in \bar{\Omega}$ and if $P = (1-\lambda)x + \lambda y$, some $0 < \lambda < 1$, then $x = y = P$. Although it is necessary and sufficient for strong convexity of a point P that all boundary curvatures be positive, it is only necessary that the boundary point be extreme. The point $P = (1, 0)$ in the boundary of the convex domain $\{(x_1, x_2) \in \mathbb{R}^2 : |x_1|^2 + |x_2|^4 < 1\}$ is extreme, but is not a point of strong convexity.

Now we explore our analytic notions of convexity. The first lemma is a technical one.

LEMMA 3.1.4 Let $\Omega \subseteq \mathbb{R}^N$ be strongly convex. Then there is a constant $C > 0$ and a defining function $\tilde{\rho}$ for Ω such that

$$\sum_{j,k=1}^{N} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) w_j w_k \geq C|w|^2, \qquad \forall P \in \partial\Omega, w \in \mathbb{R}^N. \tag{3.1.4.1}$$

Proof Let ρ be some fixed C^2 defining function for Ω. For $\lambda > 0$, define

$$\rho_\lambda(x) = \frac{\exp(\lambda \rho(x)) - 1}{\lambda}.$$

We shall select λ large in a moment. Let $P \in \partial\Omega$ and set

$$X = X_P = \left\{ w \in \mathbb{R}^N : |w| = 1 \text{ and } \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k \leq 0 \right\}.$$

Then no element of X could be a tangent vector at P; hence $X \subseteq \{w : |w| = 1 \text{ and } \sum_j \partial\rho/\partial x_j(P) w_j \neq 0\}$. Since X is defined by a nonstrict inequality, it is closed; it is, of course, also bounded. Hence X is compact and

$$\mu \equiv \min \left\{ \left| \sum_j \partial\rho/\partial x_j(P) w_j \right| : w \in X \right\}$$

is attained and is nonzero. Define

$$\lambda = \frac{-\min_{w \in X} \sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k}{\mu^2} + 1.$$

Set $\tilde{\rho} = \rho_\lambda$. Then for any $w \in \mathbb{R}^N$ with $|w| = 1$, we have (since $\exp(\rho(P)) = 1$) that

$$\begin{aligned}
\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k}(P) w_j w_k &= \sum_{j,k} \left\{ \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) + \lambda \frac{\partial \rho}{\partial x_j}(P) \frac{\partial \rho}{\partial x_k}(P) \right\} w_j w_k \\
&= \sum_{j,k} \left\{ \frac{\partial^2 \rho}{\partial x_j \partial x_k} \right\} (P) w_j w_k + \lambda \left| \sum_j \frac{\partial \rho}{\partial x_j}(P) w_j \right|^2.
\end{aligned}$$

If $w \notin X$ then this expression is positive by definition. If $w \in X$ then the expression is positive by the choice of λ. Since $\{w \in \mathbb{R}^N : |w| = 1\}$ is compact, there is thus a $C > 0$ such that

$$\sum_{j,k} \left\{ \frac{\partial^2 \tilde{\rho}}{\partial x_j \partial x_k} \right\} (P) w_j w_k \geq C, \qquad \forall w \in \mathbb{R}^N \text{ such that } |w| = 1.$$

This establishes our inequality (3.1.4.1) for $P \in \partial\Omega$ fixed and w in the unit sphere of $\mathbb{R}^N$. For arbitrary w, we set $w = |w|\hat{w}$, with $\hat{w}$ in the unit sphere. Then (3.1.4.1) holds for $\hat{w}$. Multiplying both sides of the inequality for $\hat{w}$ by $|w|^2$ and performing some algebraic manipulations gives the result for fixed P and all $w \in \mathbb{R}^N$. (In the future we shall refer to this type of argument as a "homogeneity argument.")

Finally, notice that our estimates—in particular, the existence of C—hold uniformly over points in $\partial\Omega$ near P. Since $\partial\Omega$ is compact, we see that the constant C may be chosen uniformly over all boundary points of Ω. □

Notice that the statement of the lemma has two important features: (1) that the constant C may be selected uniformly over the boundary and (2) that the inequality (3.1.4.1) holds for all $w \in \mathbb{R}^N$ (not just tangent vectors). In fact, it is impossible to arrange for anything like (3.1.4.1) to be true at a weakly convex point.

Our proof shows in fact that (3.1.4.1) is true not just for $P \in \partial\Omega$ but for P in a neighborhood of $\partial\Omega$. It is this sort of stability of the notion of strong convexity that makes it a more useful device than ordinary (weak) convexity.

PROPOSITION 3.1.5 If Ω is strongly convex, then Ω is geometrically convex.

Proof We use a connectedness argument.

Clearly $\Omega \times \Omega$ is connected. Set $S = \{(P_1, P_2) \in \Omega \times \Omega : (1-\lambda)P_1 + \lambda P_2 \in \Omega,\ \text{all } 0 < \lambda < 1\}$. Then S is plainly open and nonempty.

To see that S is closed, fix a defining function $\tilde{\rho}$ for Ω as in the lemma. If S is not closed in $\Omega \times \Omega$, then there exist $P_1, P_2 \in \Omega$ such that the function

$$t \mapsto \tilde{\rho}((1-t)P_1 + tP_2)$$

assumes an interior maximum value of 0 on $[0, 1]$. But the positive definiteness of the real Hessian of $\tilde{\rho}$ contradicts that assertion. The proof is complete. □

We gave a special proof that strong convexity implies geometric convexity simply to illustrate the utility of the strong convexity concept. It is possible to prove that an arbitrary (weakly) convex domain is geometrically convex by showing that such a domain can be written as the increasing union of strongly convex domains. However the proof is difficult and technical (the reader interested in these matters may wish to consider them after learning the techniques in the proof of Theorem 3.3.5). We thus give another proof of this fact.

PROPOSITION 3.1.6 If Ω is (weakly) convex, then Ω is geometrically convex.

Proof To simplify the proof we shall assume that Ω has at least C^3 boundary.
Assume without loss of generality that $N \geq 2$ and $0 \in \Omega$. For $\epsilon > 0$, let $\rho_\epsilon(x) = \rho(x) + \epsilon|x|^{2M}/M$ and $\Omega_\epsilon = \{x : \rho_\epsilon(x) < 0\}$. Then $\Omega_\epsilon \subseteq \Omega_{\epsilon'}$ if $\epsilon' < \epsilon$ and $\cup_{\epsilon>0}\Omega_\epsilon = \Omega$. If $M \in \mathbb{N}$ is large and ϵ is small, then Ω_ϵ is strongly convex. By Proposition 3.1.5, each Ω_ϵ is geometrically convex, so Ω is convex. □

We mention in passing that a nice treatment of convexity, from roughly the point of view presented here, appears in V. Vladimirov [1].

PROPOSITION 3.1.7 Let $\Omega \subset\subset \mathbb{R}^N$ have C^2 boundary and be geometrically convex. Then Ω is (weakly) convex.

Proof Seeking a contradiction, we suppose that for some $P \in \partial\Omega$ and some $w \in T_P(\partial\Omega)$, we have

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k = -2K < 0. \tag{3.1.7.1}$$

Suppose without loss of generality that coordinates have been selected in $\mathbb{R}^N$ so that $P = 0$ and $(0, 0, \ldots, 0, 1)$ is the unit outward normal vector to $\partial\Omega$ at P. We may further normalize the defining function ρ so that $\partial\rho/\partial x_N(0) = 1$. Let $Q = Q^t = tw + \epsilon \cdot (0, 0, \ldots, 0, 1)$, where $\epsilon > 0$ and $t \in \mathbb{R}$. Then, by Taylor's expansion,

$$\begin{aligned} \rho(Q) &= \rho(0) + \sum_{j=1}^N \frac{\partial \rho}{\partial x_j}(0) Q_j + \frac{1}{2} \sum_{j,k=1}^N \frac{\partial^2 \rho}{\partial x_j \partial x_k}(0) Q_j Q_k + o(|Q|^2) \\ &= \epsilon \frac{\partial \rho}{\partial x_N}(0) + \frac{t^2}{2} \sum_{j,k=1}^N \frac{\partial^2 \rho}{\partial x_j \partial x_k}(0) w_j w_k + \mathcal{O}(\epsilon^2) + o(t^2) \\ &= \epsilon - Kt^2 + \mathcal{O}(\epsilon^2) + o(t^2). \end{aligned}$$

Thus if $t = 0$ and $\epsilon > 0$ is small enough, then $\rho(Q) > 0$. However, for that same value of ϵ, if $|t| > \sqrt{2\epsilon/K}$, then $\rho(Q) < 0$. This contradicts the definition of geometric convexity. □

Remark: The reader can already see in the proof of the proposition how useful the quantitative version of convexity can be.

The assumption that $\partial\Omega$ be C^2 is not very restrictive, for convex functions of one variable are twice differentiable almost everywhere (see A. Zygmund [1]). On the other hand, C^2 smoothness of the boundary is essential for our approach to the subject. □

Exercise for the Reader

If $\Omega \subseteq \mathbb{R}^N$ is a domain, then the *closed convex hull* of Ω is defined to be the closure of the set $\{\sum_{j=1}^m \lambda_j s_j : s_j \in \Omega, m \in \mathbb{N}, \lambda_j \geq 0, \sum \lambda_j = 1\}$.

Assume in the following problems that $\bar{\Omega} \subseteq \mathbb{R}^N$ is closed, bounded, and convex. Assume that Ω has C^2 boundary.

1. Prove that $\bar{\Omega}$ is the closed convex hull of its extreme points (this result is usually referred to as the *Krein-Milman theorem* and is true in much greater generality).
2. Let $P \in \partial\Omega$ be extreme. Let $\mathbf{p} = P + T_P(\partial\Omega)$ be the geometric tangent affine hyperplane to the boundary of Ω that passes through P. Show by an example that it is not necessarily the case that $\mathbf{p} \cap \bar{\Omega} = \{P\}$.
3. Prove that if Ω_0 is *any* bounded domain with C^2 boundary, then there is a relatively open subset U of $\partial\Omega_0$ such that U is strongly convex. (*Hint:* Fix $x_0 \in \Omega_0$ and choose $P \in \partial\Omega_0$ that is as far as possible from x_0).
4. If Ω is a convex domain, then the Minkowski functional (see S. R. Lay [1]) gives a convex defining function for Ω.

Our goal now is to pass from convexity to a complex-analytic analogue of convexity. We shall first express the differential condition for convexity in complex notation. Then we shall isolate that portion of the complexified expression that is invariant under biholomorphic mappings. This invariant version of convexity will be the focus of much of our study for the remainder of the book. Because of its centrality we have gone to extra trouble to put all these ideas into context.

Now fix $\Omega \subset\subset \mathbb{C}^n$ with C^2 boundary and assume that $\partial\Omega$ is convex at $P \in \partial\Omega$. If $w \in \mathbb{C}^n$, then the complex coordinates for w are, of course,

$$w = (w_1, \ldots, w_n) = (\xi_1 + i\eta_1, \ldots, \xi_n + i\eta_n).$$

Then it is natural to (geometrically) identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the map

$$(\xi_1 + i\eta_1, \ldots, \xi_n + i\eta_n) \longleftrightarrow (\xi_1, \eta_1, \ldots, \xi_n, \eta_n).$$

Similarly, we identify $z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n$ with $(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$. (Strictly speaking, $\mathbb{C}^n$ is $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$. Then one equips $\mathbb{C}^n$ with a linear map J, called the *complex structure tensor*, which mediates between the algebraic operation of multiplying by i and the geometric mapping $(\xi_1, \eta_1, \ldots, \xi_n, \eta_n) \mapsto (-\eta_1, \xi_1, \ldots, -\eta_n, \xi_n)$. In this book it would be both tedious and unnatural to indulge in these niceties. In other contexts they are essential. See R. O. Wells [2] for a thorough treatment of this matter.) If ρ is a defining function for Ω that is C^2 near P, then the condition that $w \in T_P(\partial\Omega)$ is

$$\sum_j \frac{\partial\rho}{\partial x_j}\xi_j + \sum_j \frac{\partial\rho}{\partial y_j}\eta_j = 0.$$

In complex notation we may write this equation as

$$\frac{1}{2}\sum_j\left[\left(\frac{\partial}{\partial z_j}+\frac{\partial}{\partial \bar{z}_j}\right)\rho(P)\right](w_j+\bar{w}_j)$$
$$+\frac{1}{2}\sum_j\left[\left(\frac{1}{i}\right)\left(\frac{\partial}{\partial \bar{z}_j}-\frac{\partial}{\partial z_j}\right)\rho(P)\right]\left(\frac{1}{i}\right)(w_j-\bar{w}_j)=0.$$

But this is the same as

$$2\mathrm{Re}\left(\sum_j\frac{\partial\rho}{\partial z_j}(P)w_j\right)=0.$$

The space of vectors w that satisfy this last equation is not closed under multiplication by i, and hence is not a natural object of study for our purposes. Instead, we restrict attention in the following discussion to vectors $w \in \mathbb{C}^n$ that satisfy

$$\sum_j\frac{\partial\rho}{\partial z_j}(P)w_j=0.$$

The collection of all such vectors is termed the *complex tangent space* to $\partial\Omega$ at P and is denoted by $\mathcal{T}_P(\partial\Omega)$. Clearly $\mathcal{T}_P(\partial\Omega)\subseteq T_P(\partial\Omega)$; indeed, the complex tangent space is a proper real subspace of the ordinary (real) tangent space. The reader should check that $\mathcal{T}_P(\partial\Omega)$ is the largest complex subspace of $T_P(\partial\Omega)$ in the following sense: If S is a real linear subspace of $T_P(\partial\Omega)$ that is closed under multiplication by i, then $S\subseteq\mathcal{T}_P(\partial\Omega)$. In particular, when $n=1,\Omega\subseteq\mathbb{C}^n$, and $P\in\partial\Omega$ then $\mathcal{T}_P(\partial\Omega)=\{0\}$. At some level, this last fact explains many of the differences between the functions theories of one and several complex variables. Now we turn attention to the convexity condition.

The convexity condition on tangent vectors is

$$\begin{aligned}0 \;\le\; & \sum_{j,k=1}^n\frac{\partial^2\rho}{\partial x_j\partial x_k}(P)\xi_j\xi_k\\ & +2\sum_{j,k=1}^n\frac{\partial^2\rho}{\partial x_j\partial y_k}(P)\xi_j\eta_k+\sum_{j,k=1}^n\frac{\partial^2\rho}{\partial y_j\partial y_k}(P)\eta_j\eta_k\\ = \;& \frac{1}{4}\sum_{j,k=1}^n\left(\frac{\partial}{\partial z_j}+\frac{\partial}{\partial \bar{z}_j}\right)\left(\frac{\partial}{\partial z_k}+\frac{\partial}{\partial \bar{z}_k}\right)\rho(P)(w_j+\bar{w}_j)(w_k+\bar{w}_k)\\ & +2\cdot\frac{1}{4}\sum_{j,k=1}^n\left(\frac{\partial}{\partial z_j}+\frac{\partial}{\partial \bar{z}_j}\right)\left(\frac{1}{i}\right)\left(\frac{\partial}{\partial \bar{z}_k}-\frac{\partial}{\partial z_k}\right)\rho(P)\end{aligned}$$

$$\begin{aligned}
&\times(w_j+\bar{w}_j)\left(\frac{1}{i}\right)(w_k-\bar{w}_k)\\
&+\frac{1}{4}\sum_{j,k=1}^{n}\left(\frac{1}{i}\right)\left(\frac{\partial}{\partial\bar{z}_j}-\frac{\partial}{\partial z_j}\right)\left(\frac{1}{i}\right)\left(\frac{\partial}{\partial\bar{z}_k}-\frac{\partial}{\partial z_k}\right)\rho(P)\\
&\times\left(\frac{1}{i}\right)(w_j-\bar{w}_j)\left(\frac{1}{i}\right)(w_k-\bar{w}_k)\\
=\;&\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial z_k}(P)w_jw_k+\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial\bar{z}_j\partial\bar{z}_k}(P)\bar{w}_j\bar{w}_k\\
&+2\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(P)w_j\bar{w}_k\\
=\;&2\operatorname{Re}\left(\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial z_k}(P)w_jw_k\right)+2\sum_{j,k=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\bar{z}_k}(P)w_j\bar{w}_k.
\end{aligned}$$

(This formula could also have been derived by examining the complex form of Taylor's formula—see Exercise 35 at the end of Chapter 1.) We see that the real Hessian, when written in complex coordinates, decomposes rather naturally into two Hessian-like expressions. Our next task is to see that the first of these does not transform canonically under biholomorphic mappings but the second one does. We shall thus dub the second quadratic expression "the complex Hessian" of ρ. It will also be called the "Levi form" of the domain Ω. This form will be the object of our considerable attention for the remainder of this book.

The Riemann mapping theorem tells us, in part, that the unit disc is biholomorphic to any simply connected smoothly bounded planar domain. Since many of these domains are not convex, we see easily that biholomorphic mappings do not preserve convexity (an explicit example of this phenomenon is the mapping $\phi : D \to \phi(D), \phi(\zeta) = (\zeta + 2)^4$). We wish now to understand analytically where the failure lies. So let $\Omega \subset\subset \mathbb{C}^n$ be a convex domain with C^2 boundary. Let U be a neighborhood of $\bar{\Omega}$ and $\rho : U \to \mathbb{R}$ a defining function for Ω. Assume that $\Phi : U \to \mathbb{C}^n$ is biholomorphic onto its image and define $\Omega' = \Phi(\Omega)$. Hopf's lemma (Exercise 22 at the end of Chapter 1; the proof shows that Hopf's lemma is valid for subharmonic and hence for plurisubharmonic functions) guarantees that $\rho' \equiv \rho \circ \Phi^{-1}$ is a defining function for Ω'. Finally fix a point $P \in \partial\Omega$ and corresponding point $P' \equiv \Phi(P) \in \partial\Omega'$. If $w \in \mathcal{T}_P(\partial\Omega)$, then

$$w' = \left(\sum_{j=1}^{n}\frac{\partial\Phi_1(P)}{\partial z_j}w_j,\ldots,\sum_{j=1}^{n}\frac{\partial\Phi_n(P)}{\partial z_j}w_j\right) \in \mathcal{T}_{P'}(\partial\Omega').$$

Let the complex coordinates on $\Phi(U)$ be $z_1{}', \ldots, z_n'$. Our task is to write the expression determining convexity,

$$2 \operatorname{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P) w_j w_k\right) + 2\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k, \tag{3.1.8}$$

in terms of the $z_j{}'$ and the $w_j{}'$. But

$$\begin{aligned} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P) &= \frac{\partial}{\partial z_j} \sum_{\ell=1}^{n} \frac{\partial \rho'}{\partial z'_\ell} \frac{\partial \Phi_\ell}{\partial z_k} \\ &= \sum_{\ell,m=1}^{n} \left\{ \frac{\partial^2 \rho'}{\partial z'_\ell \partial z'_m} \frac{\partial \Phi_\ell}{\partial z_k} \frac{\partial \Phi_m}{\partial z_j} \right\} + \sum_{\ell=1}^{n} \left\{ \frac{\partial \rho'}{\partial z'_\ell} \frac{\partial^2 \Phi_\ell}{\partial z_j \partial z_k} \right\}, \\ \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) &= \frac{\partial}{\partial z_j} \sum_{\ell=1}^{n} \frac{\partial \rho'}{\partial \bar{z}'_\ell} \frac{\partial \bar{\Phi}_\ell}{\partial \bar{z}_k} = \sum_{\ell,m=1}^{n} \frac{\partial^2 \rho'}{\partial z'_m \partial \bar{z}'_\ell} \frac{\partial \Phi_m}{\partial z_j} \frac{\partial \bar{\Phi}_\ell}{\partial \bar{z}_k}. \end{aligned}$$

Therefore,

$$(3.1.8) = \underbrace{2 \operatorname{Re}\left\{ \sum_{\ell,m=1}^{n} \frac{\partial^2 \rho'}{\partial z'_\ell \partial z'_m} w'_\ell w'_m + \sum_{j,k=1}^{n} \sum_{\ell=1}^{n} \frac{\partial \rho'}{\partial z'_\ell} \frac{\partial^2 \Phi_\ell}{\partial z_j \partial z_k} w_j w_k \right\}}_{\text{nonfunctorial}}$$

$$\underbrace{+2 \sum_{\ell,m=1}^{n} \frac{\partial^2 \rho'}{\partial z'_m \partial \bar{z}'_\ell} w'_m \bar{w}'_\ell}_{\text{functorial}}.$$

So we see that the part of the quadratic form characterizing convexity that is preserved under biholomorphic mappings is

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k.$$

DEFINITION 3.1.8 Let $\Omega \subseteq \mathbb{C}^n$ be a domain with C^2 boundary and let $P \in \partial\Omega$. Let ρ be a C^2 defining function for Ω. We say that $\partial\Omega$ is *Levi pseudoconvex* at P if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq 0, \qquad \forall w \in \mathcal{T}_P(\partial\Omega). \tag{3.1.8.1}$$

The expression on the left side of (3.1.8.1) is called the *Levi form.* The point P is said to be *strongly (or strictly) Levi pseudoconvex* if the expression on the left side of (3.1.8.1) is positive whenever $w \neq 0, w \in \mathcal{T}_P(\partial\Omega)$. A domain is called *Levi pseudoconvex* (resp. *strongly Levi pseudoconvex*) if all its boundary points are pseudoconvex (resp. strongly Levi pseudoconvex).

The reader may check that the definition of pseudoconvexity is independent of the choice of defining function for the domain in question.

The collection of Levi pseudoconvex domains is, in a local sense to be made precise later, the smallest class of domains that contains the convex domains and is closed under increasing union and biholomorphic mappings.

PROPOSITION 3.1.9 If $\Omega \subseteq \mathbb{C}^n$ is a domain with C^2 boundary and if $P \in \partial\Omega$ is a point of convexity, then P is also a point of pseudoconvexity.

Proof Let ρ be a defining function for Ω. Let $w \in \mathcal{T}_P(\partial\Omega)$. Then iw is also in $\mathcal{T}_P(\partial\Omega)$. If we apply the convexity hypothesis to w at P, we obtain

$$2\,\mathrm{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P) w_j w_k\right) + 2\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq 0.$$

However, if we apply the convexity condition to iw at P, we obtain

$$-2\,\mathrm{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P) w_j w_k\right) + 2\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq 0.$$

Adding these two inequalities we find that

$$4\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq 0;$$

hence $\partial\Omega$ is Levi pseudoconvex at P. □

The converse of this lemma is false. For instance, any product of smooth domains $\Omega_1 \times \Omega_2 \subseteq \mathbb{C}^2$ is Levi pseudoconvex at boundary points that are smooth (for instance, off the distinguished boundary $\partial\Omega_1 \times \partial\Omega_2$). From this observation a smooth example may be obtained simply by rounding off the product domain near its distinguished boundary. The reader should carry out the details of these remarks as an exercise.

There is no elementary geometric way to think about pseudoconvex domains. The collection of convex domains forms an important subclass, but by no means a representative subclass. As recently as 1972 it was conjectured that a pseudoconvex point $P \in \partial\Omega$ has the property that there is a holomorphic change of coordinates Φ on a neighborhood U of P such that $\Phi(U \cap \partial\Omega)$ is convex. This conjecture is false (see J. J. Kohn and L. Nirenberg [1]). In fact it is not known which pseudoconvex boundary points are "convexifiable."

The definition of Levi pseudoconvexity can be motivated by analogy with the real variable definition of convexity. However, we feel that the calculations above, which we learned from J. J. Kohn, provide the most palpable means of establishing the importance of the Levi form.

We conclude this discussion by noting that pseudoconvexity is not an interesting condition in one complex dimension because the complex tangent space to the boundary of a domain is always empty. Any domain in the complex plane is vacuously pseudoconvex.

3.1.2 Convexity with Respect to a Family of Functions

Let $\Omega \subseteq \mathbb{R}^N$ be a domain and let $\mathcal{F}$ be a family of real-valued functions on Ω (we do not assume in advance that $\mathcal{F}$ is closed under any algebraic operations, although often in practice it will be). Let K be a compact subset of Ω. Then the *convex hull of* K *in* Ω *with respect to* $\mathcal{F}$ is defined to be

$$\hat{K}_{\mathcal{F}} \equiv \left\{ x \in \Omega : f(x) \leq \sup_{t \in K} f(t) \text{ for all } f \in \mathcal{F} \right\}.$$

We sometimes denote this hull by $\hat{K}$ when the family $\mathcal{F}$ is understood or when no confusion is possible. We say that Ω is *convex* with respect to $\mathcal{F}$ provided $\hat{K}_{\mathcal{F}}$ is compact in Ω whenever K is. When the functions in $\mathcal{F}$ are complex-valued, then $|f|$ replaces f in the definition of $\hat{K}_{\mathcal{F}}$.

PROPOSITION 3.1.10 Let $\Omega \subset\subset \mathbb{R}^N$ and let $\mathcal{F}$ be the family of real linear functions. Then Ω is convex with respect to $\mathcal{F}$ if and only if Ω is geometrically convex.

Proof The proof is left as an exercise. Use the classical definition of convexity at the beginning of the section. □

PROPOSITION 3.1.11 Let $\Omega \subset\subset \mathbb{R}^N$ be any domain. Let $\mathcal{F}$ be the family of continuous functions on Ω. Then Ω is convex with respect to $\mathcal{F}$.

Proof If $K \subset\subset \Omega$ and $x \notin K$, then the function $F(t) = 1/(1 + |x - t|)$ is continuous on Ω. Notice that $f(x) = 1$ and $|f(k)| < 1$ for all $k \in K$. Thus $x \notin \hat{K}_{\mathcal{F}}$. Therefore, $\hat{K}_{\mathcal{F}} = K$ and Ω is convex with respect to $\mathcal{F}$. □

PROPOSITION 3.1.12 Let $\Omega \subseteq \mathbb{C}$ be an open set and let $\mathcal{F}$ be the family of all functions holomorphic on Ω. Then Ω is convex with respect to $\mathcal{F}$.

Proof First suppose that Ω is bounded. Let $K \subset\subset \Omega$. Let r be the Euclidean distance of K to the complement of Ω. Then $r > 0$. Suppose that $w \in \Omega$ is of distance less than r from $\partial\Omega$. Choose $w' \in \partial\Omega$ such that $|w-w'| = \text{dist}(w, {}^c\Omega)$. Then the function $f(\zeta) = 1/(\zeta - w')$ is holomorphic on Ω and $|f(w)| > \sup_{\zeta \in K} |f(\zeta)|$. Hence $w \notin \hat{K}_{\mathcal{F}}$, so $\hat{K}_{\mathcal{F}} \subset\subset \Omega$. Therefore, Ω is convex with respect to $\mathcal{F}$.

In case Ω is unbounded, we take a large disc $D(0, R)$ containing K and notice that $\hat{K}_{\mathcal{F}}$ with respect to Ω is equal to $\hat{K}_{\mathcal{F}}$ with respect to $\Omega \cap D(0, R)$, which by the first part of the proof is relatively compact. □

Exercise for the Reader

Prove that, in the last proposition, if the family $\mathcal{F}$ is replaced by the family $\mathcal{G}$ of bounded holomorphic functions, then not every Ω will be convex with respect to $\mathcal{G}$ (however your example will not have smooth boundary). What about the family $\mathcal{H}$ of square integrable holomorphic functions?

The reader should consider carefully why the argument in the last proposition breaks down in $\mathbb{C}^n, n \geq 2$. The reason is that, in higher dimensions, the zeros of a nonconstant holomorphic function are never isolated. The fact that holomorphic functions of one variable do have isolated zeros made it very easy to construct the required function f. But in $\mathbb{C}^2$, a holomorphic function that vanishes at $w' \in \partial\Omega$ might also perforce vanish at points inside the domain as well.

In fact, it is the notion of pseudoconvexity that helps to detect when the zeros of a holomorphic function that vanishes at a boundary point of a domain can be kept outside the domain. This is a subject that we explore in detail as the book develops.

With respect to the exercise, we note that in $\mathbb{C}^2$, there is a bounded, topologically trivial domain Ω with smooth boundary such that Ω is convex with respect to the family of all holomorphic functions but not with respect to all bounded holomorphic functions (see N. Sibony [1, 3]). A variant of this example will be considered in Exercise 7 at the end of Chapter 8.

3.1.3 Concluding Remarks

The discussion thus far in this chapter has shown that convexity for domains and convexity for functions are closely related concepts. We now develop the latter notion a bit further.

Classically, a real-valued function f on a convex domain Ω is called *convex* if, whenever $P, Q \in \Omega$ and $0 \leq \lambda \leq 1$, we have $f((1-\lambda)P + \lambda Q) \leq (1-\lambda)f(P) + \lambda f(Q)$. A C^2 function f is convex according to this definition if and only if the matrix $\left(\partial^2 f/\partial x_j \partial x_k\right)_{j,k=1}^{N}$ is positive semidefinite at each point of the domain of f. The function is convex *at a point* if this Hessian matrix is positive semidefinite at that point. It is *strongly* (or strictly) *convex* at a point of its domain if the matrix is strictly positive definite at that point. Of course, the function is called strictly convex if it is such at each point of its domain.

Now let $\Omega \subseteq \mathbb{R}^N$ be any domain. A function $\phi : \Omega \to \mathbb{R}$ is called an *exhaustion function* for Ω if, for any $c \in \mathbb{R}$, the set $\Omega_c \equiv \{x \in \Omega : \phi(x) \leq c\}$ is relatively compact in Ω. It is a fact (not easy to prove) that Ω is convex if and only if it possesses a convex exhaustion function, and that is true if and only if it possesses a strictly convex exhaustion function. In Section 3.3 we shall present the necessary tools for proving a result such as this.

We close this discussion of convexity with a geometric characterization of the property. We shall, later in the book, refer to this as the "affine sphere" characterization. First, if $\Omega \subseteq \mathbb{R}^N$ is a domain and I is a closed one-dimensional segment lying in Ω, then the boundary ∂I is the set consisting of the two endpoints of I. Now the domain Ω is convex if and only if whenever $\{I_j\}_{j=1}^{\infty}$ is a collection of segments in Ω and $\{\partial I_j\}$ is relatively compact in Ω, then so is $\{I_j\}$. This is little more than a restatement of the classical definition of geometric convexity. We invite the reader to supply the details.

All our approaches to convexity will prove useful in later parts of the book. Each of them will have an analogue in the complex analytic setting, and each will be part of the arsenal of tools that we use to characterize domains of holomorphy. In fact, one of the main goals of the first five chapters of this book is to prove the following equivalence (here $\Omega \subseteq \mathbb{C}^n$ is a domain with C^2 boundary and $\mathcal{F}$ is the family of holomorphic functions on Ω):

Ω is a domain of holomorphy. $\Longleftrightarrow$

Ω is convex with respect to $\mathcal{F}$. $\Longleftrightarrow$

Ω is Levi pseudoconvex. $\Longleftrightarrow$

Ω has a C^∞ strictly psh exhaustion function. $\Longleftrightarrow$

The equation $\bar{\partial} u = f$ can be solved on Ω for every $\bar{\partial}$-closed (p,q) form f on Ω. $\Longleftrightarrow$

Whenever $\{\delta_j\}_{j=1}^{\infty} \subseteq \Omega$ is a family of closed analytic discs (see Section 3.2) such that $\{\partial \delta_j\}$ is relatively compact in Ω, then $\{\delta_j\}$ is also relatively compact in Ω.

The hardest part of these equivalences is that a Levi pseudoconvex domain is a domain of holomorphy. This implication is known as the *Levi problem* and was solved completely for domains in $\mathbb{C}^n$, all n, only in the mid-1950s. Some generalizations of the problem to complex manifolds and analytic spaces remain open. An informative survey is Y. T. Siu [1]. We shall use the technique of the $\bar{\partial}$ equation to attack the Levi problem; that is, we first prove that the $\bar{\partial}$ equation can always be solved on a Levi pseudoconvex domain. Then we use this tool to show that a Levi pseudoconvex domain is a domain of holomorphy.

The next section collects a number of geometric properties of pseudoconvex domains. Although some of these properties are not needed for a while, it is appropriate to treat these properties all in one place.

3.2 Convexity and Pseudoconvexity

Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with C^2 boundary. If $P \in \partial\Omega$, then P is a point of (weak) *Levi pseudoconvexity* if the Levi form is positive semi-definite on the space of $w \in \mathcal{T}_P(\Omega)$. Explicitly, $P \in \partial\Omega$ is a point of Levi pseudoconvexity for $\Omega = \{z \in \mathbb{C}^N : \rho(z) < 0\}$ if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq 0$$

for all $w \in \mathbb{C}^N$ that satisfy

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P) w_j = 0.$$

The point P is a point of *strong* (or *strict*) pseudoconvexity if the Levi form at P is positive definite for some choice of defining function and $w \in \mathcal{T}_P(\partial\Omega)$. One checks that these definitions are in fact independent of the choice of defining function. The domain Ω is said to be *Levi pseudoconvex* (resp. *strictly or strongly Levi pseudoconvex*) if every $P \in \partial\Omega$ is a point of Levi pseudoconvexity (resp. strict or strong Levi pseudoconvexity).

A line-by-line imitation of the proof of Lemma 3.1.4 yields the next result.

PROPOSITION 3.2.1 If Ω is strongly pseudoconvex, then Ω has a defining function $\tilde{\rho}$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq C|w|^2$$

for all $P \in \partial\Omega$, all $w \in \mathbb{C}^n$.

By continuity of the second derivatives of $\tilde{\rho}$, the inequality in the proposition must in fact persist for all z in a neighborhood of $\partial\Omega$. In particular, if $P \in \partial\Omega$ is a point of strong pseudoconvexity, then so are all nearby boundary points. The analogous assertion for weakly pseudoconvex points is false.

EXAMPLE Let $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^4 < 1\}$. Then $\rho(z_1, z_2) = -1 + |z_1|^2 + |z_2|^4$ is a defining function for Ω and the Levi form applied to (w_1, w_2) is $|w_1|^2 + 4|z_2|^2|w_2|^2$. Thus $\partial\Omega$ is strongly pseudoconvex except at boundary points where $|z_2|^2 = 0$, and the tangent vectors w satisfy $w_1 = 0$. Of course, these are just the boundary points of the form $(e^{i\theta}, 0)$. The domain is (weakly) Levi pseudoconvex at these exceptional points. □

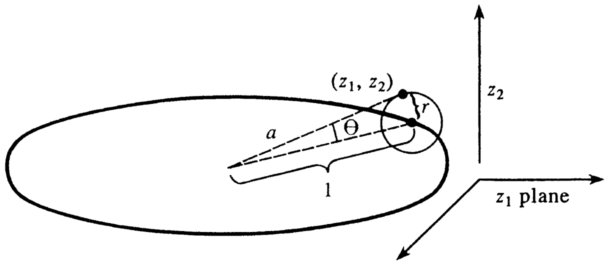

FIGURE 3.1

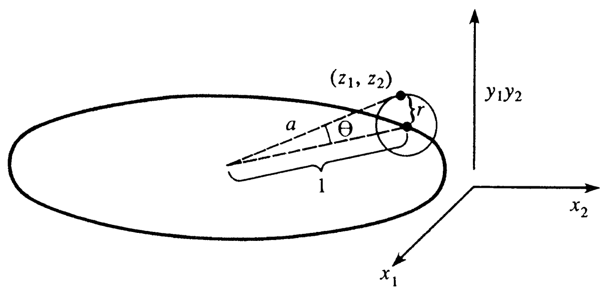

FIGURE 3.2

EXAMPLE Let Ω_1 be the "solid torus" in $\mathbb{C}^2$ whose major circle of rotation (of radius 1) is independent of z_2 and whose minor circle of rotation (of radius r) is independent of z_1 (see Figure 3.1). This region is described by the equation

$$a^2 + 1^2 - 2 \cdot a \cdot 1 \cdot \cos\theta < r^2;$$

that is, the domain has defining function

$$\rho_1(z_1, z_2) = |z_1|^2 + |z_2|^2 + 1 - 2|z_1| - r^2.$$

The reader may calculate that the domain is strongly pseudoconvex for $0 < r < 1$.

On the other hand, let Ω_2 be the "solid torus" in $\mathbb{C}^2$ whose major circle of rotation (of radius 1) lies in the $x_1 - x_2$ plane and whose minor circle of rotation (of radius r) is in the $y_1 - y_2$ plane (see Figure 3.2). This region is described by the equation

$$a^2 + 1^2 - 2 \cdot a \cdot 1 \cdot \cos\theta < r^2;$$

that is, the domain has defining function

$$\rho_1(z_1, z_2) = |z_1|^2 + |z_2|^2 + 1 - 2\sqrt{x_1^2 + x_2^2} - r^2.$$

The reader may calculate the Levi form of Ω_2 and determine that Ω_2 is pseudoconvex *only if* $r \leq 1/2$. It is strongly pseudoconvex if $0 < r < 1/2$.

The two tori described here are isometric in Euclidean geometry. But they are *not* biholomorphic. They are imbedded into complex Euclidean space in such a fashion that they are different from the point of view of complex analysis. □

We see from the last example that pseudoconvexity describes something more (and less) than classical geometric properties of a domain. We will learn later that certain cohomology groups of Ω with coefficients in $\mathbb{C}$ must vanish when Ω is pseudoconvex. Also, pseudoconvexity can be characterized by a condition similar to the classical characterization of convexity (presented in Section 3.1) by families of imbedded segments; for pseudoconvexity we replace segments by holomorphically imbedded discs. However, it is important to realize that there is no simple geometric description of pseudoconvex points. Weakly pseudoconvex points are far from being well understood at this time. Matters are much clearer for strongly pseudoconvex points.

LEMMA 3.2.2 (Narasimhan) Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with C^2 boundary. Let $P \in \partial\Omega$ be a point of strong pseudoconvexity. Then there is a neighborhood $U \subseteq \mathbb{C}^n$ of P and a biholomorphic mapping Φ on U such that $\Phi(U \cap \partial\Omega)$ is strongly convex.

Proof By Proposition 3.2.1 there is a defining function $\tilde{\rho}$ for Ω such that

$$\sum_{j,k} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k \geq C|w|^2$$

for all $w \in \mathbb{C}^n$. By a rotation and translation of coordinates, we may assume that $P = 0$ and that $\nu = (1, 0, \ldots, 0)$ is the unit outward normal to $\partial\Omega$ at P. The second-order Taylor expansion of $\tilde{\rho}$ about $P = 0$ is given by

$$\begin{aligned}
\tilde{\rho}(w) &= \tilde{\rho}(0) + \sum_{j=1}^{n} \frac{\partial \tilde{\rho}}{\partial z_j}(P) w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial z_k}(P) w_j w_k \\
&\quad + \sum_{j=1}^{n} \frac{\partial \tilde{\rho}}{\partial \bar{z}_j}(P) \bar{w}_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial \bar{z}_j \partial \bar{z}_k}(P) \bar{w}_j \bar{w}_k \\
&\quad + \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k + o(|w|^2) \\
&= 2 \operatorname{Re} \left\{ \sum_{j=1}^{n} \frac{\partial \tilde{\rho}}{\partial z_j}(P) w_j + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial z_k}(P) w_j w_k \right\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k + o(|w|^2) \\
&= 2\,\mathrm{Re}\left\{ w_1 + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial z_k}(P) w_j w_k \right\} \\
&+\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(P) w_j \bar{w}_k + o(|w|^2)
\end{aligned}
\tag{3.2.2.1}
$$

by our normalization $\nu = (1, 0, \ldots, 0)$.

Define the mapping $w = (w_1, \ldots, w_n) \mapsto w' = (w'_1, \ldots, w'_n)$ by

$$
\begin{aligned}
w'_1 &= \Phi_1(w) &&= w_1 + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial z_k}(P) w_j w_k \\
w'_2 &= \Phi_2(w) &&= w_2 \\
&\vdots && \vdots \\
w'_n &= \Phi_n(w) &&= w_n .
\end{aligned}
$$

By the implicit function theorem, we see that for w sufficiently small this is a well-defined invertible holomorphic mapping on a small neighborhood W of $P = 0$. Then equation (3.2.2.1) tells us that, in the coordinate w', the defining function becomes

$$
\hat{\rho}(w') = 2\,\mathrm{Re}\, w'_1 + \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_j \partial \bar{z}'_k}(P) w'_j \bar{w}'_k + o(|w'|^2).
$$

Thus the real Hessian at P of the defining function $\hat{\rho}$ is precisely the Levi form; and the latter is positive definite by our hypothesis. Hence the boundary of $\Phi(W \cap \Omega)$ is strictly convex at $\Phi(P)$. By the continuity of the second derivatives of $\hat{\rho}$, we may conclude that the boundary of $\Phi(W \cap \Omega)$ is strictly convex in a neighborhood V of $\Phi(P)$. We now select $U \subseteq W$ a neighborhood of P such that $\Phi(U) \subseteq V$ to complete the proof. □

By a very ingenious (and complicated) argument, J. E. Fornæss [1] has refined Narasimhan's lemma in the following manner.

THEOREM 3.2.3 (Fornæss) Let $\Omega \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with C^2 boundary. Then there is an integer $n' > n$, a strongly convex domain $\Omega' \subseteq \mathbb{C}^{n'}$, a neighborhood $\hat{\Omega}$ of $\bar{\Omega}$, and a one-to-one imbedding $\Phi : \hat{\Omega} \to \mathbb{C}^{n'}$