CHAPTER 2

Differential Forms

In this chapter, we shall define the leading character of this book, the differential forms on differentiable manifolds.

Differential forms have two main roles. One is that they describe various system of partial differential equations on manifolds, and, ever since the pioneering work by Pfaff in the 18th and 19th centuries, they have played an important role in analysis. The other is that they are used to express various geometric structures on manifolds. By applying appropriate operations on those differential forms, various kind of differential forms are induced, and by integrating them on manifolds, certain geometric "invariants" are obtained. These invariants are quantities that reflect the global structure of manifolds, and are very important– in fact, indispensable – in the study of manifolds.

The above two roles of differential forms are deeply related to each other, rather than independent. However, in this book, keeping mainly the second role in mind, we shall introduce differential forms. That is, we consider differential forms to be something "which should be integrated on manifolds".

2.1. Definition of differential forms

(a) Differential forms on $\mathbb{R}^n$.

We start with differential forms on $\mathbb{R}^n$, for the sake of simplicity.

Recall that if an associative product is defined on a vector space Λ over the real number field $\mathbb{R}$ so that a ring structure is given and for arbitrary $a \in \mathbb{R}$ and $\lambda, \mu \in \Lambda$ the condition

$$a(\lambda\mu) = (a\lambda)\mu = \lambda(a\mu)$$

is satisfied, then Λ is called an **algebra** over $\mathbb{R}$ (Definition 1.23). An algebra generated by $dx_1, \cdots, dx_n$ over $\mathbb{R}$ with unity 1, that satisfies the equation

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \tag{2.1}$$

for arbitrary i, j, is denoted by Λ_n^*. Here $\wedge$ is a symbol that stands for the product of this algebra. We call Λ_n^* the **exterior algebra** generated by $dx_1, \cdots, dx_n$. By (2.1), we see that $dx_i \wedge dx_i = 0$ for arbitrary i. By taking the degree of dx_i to be 1, for each monomial of Λ_n^* the degree is defined. For example, the degree of $dx_1 \wedge dx_2 \wedge dx_3$ is 3. If we denote by Λ_n^k the set of all linear combinations of monomials of degree k, the direct sum decomposition

$$\Lambda_n^* = \bigoplus_{k=0}^{n} \Lambda_n^k = \Lambda_n^0 \oplus \Lambda_n^1 \oplus \cdots \oplus \Lambda_n^n$$

holds. It is easy to see that as a basis of Λ_n^k we can take

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n, \tag{2.2}$$

and hence $\dim \Lambda_n^k = \binom{n}{k}$. Also if $k > n$, then $\Lambda_n^k = 0$ and $\dim \Lambda_n^* = 2^n$.

A linear combination

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}(x_1, \cdots, x_n) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

of each element of (2.2) with C^∞ functions on $\mathbb{R}^n$ as coefficients is called a degree k **differential form** on $\mathbb{R}^n$, or simply a k-**form**. The above description is sometimes simply denoted by

$$\sum_I f_I(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

We usually express differential froms by Greek letters. We denote the set of all k-forms on $\mathbb{R}^n$ by $\mathcal{A}^k(\mathbb{R}^n)$. More precisely,

$$\mathcal{A}^k(\mathbb{R}^n) = \{\omega : \mathbb{R}^n \to \Lambda_n^k;\ C^\infty \text{map}\}$$

or

$$\mathcal{A}^k(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \otimes \Lambda_n^k.$$

Collecting differential forms of each degree, we can consider the algebra of all differential forms on $\mathbb{R}^n$,

$$\mathcal{A}^*(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \mathcal{A}^k(\mathbb{R}^n).$$

In particular, $\mathcal{A}^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$. That is, differential forms of degree 0 are simply C^∞ functions. The product $\omega \wedge \eta \in \mathcal{A}^{k+l}(\mathbb{R}^n)$ of a k-form $\omega \in \mathcal{A}^k(\mathbb{R}^n)$ and an l-form $\eta \in \mathcal{A}^l(\mathbb{R}^n)$ is defined by

$$\omega \wedge \eta = \sum_{I,J} f_I g_J \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}$$

if they are expressed as

$$\omega = \sum_I f_I(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \quad \eta = \sum_J g_J(x) dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

We call this the **exterior product** of ω and η.

In the above description, if we replace $\mathbb{R}^n$ by an open set U in $\mathbb{R}^n$, we can consider the algebra $\mathcal{A}^*(U)$ of all differential forms on U.

EXAMPLE 2.1. Put $U = \mathbb{R}^2 - \{0\}$. Then,

$$\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

is a 1-form on U. However it is not a 1-form on $\mathbb{R}^2$, because it is not defined at the origin.

The **exterior differentiation**, which is an important operation applied to differential forms, is a linear map

$$d : \mathcal{A}^k(\mathbb{R}^n) \to \mathcal{A}^{k+1}(\mathbb{R}^n),$$

defined as follows. That is, for $\omega = f(x_1, \cdots, x_n)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, let

$$d\omega = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x)\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{2.3}$$

For a function $f \in \mathcal{A}^0(\mathbb{R}^n)$ on $\mathbb{R}^n$, its exterior differentiation $df \in \mathcal{A}^1(\mathbb{R}^n)$ is $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ and is equal to so-called total differential. For practice, let ω be the 1-form in Example 2.1; if we calculate its exterior differentiation $d\omega$ by definition, we have

$$d\omega = \frac{y^2 - x^2}{(x^2+y^2)^2}\, dy \wedge dx + \frac{y^2 - x^2}{(x^2+y^2)^2}\, dx \wedge dy = 0.$$

LEMMA 2.2. *If we repeatedly operate the exterior differentiation twice, it is identically* 0. *That is,* $d \circ d = 0$.

PROOF. If we operate d again on $d\omega$ in (2.3), we have

$$d(d\omega) = \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\partial^2 f}{\partial x_l \partial x_j}\, dx_l \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Then the facts that the order 2 partial differentiation with respect to x_j and x_l does not depend on the order and $dx_l \wedge dx_j = -dx_j \wedge dx_l$ immediately imply $d(d\omega) = 0$. ∎

A differential form ω such that $d\omega = 0$ is called a **closed form**, and a differential form η that can be written $\eta = d\omega$ for some ω is called an **exact form**. The above Lemma 2.2 claims that exact forms are always closed forms. Conversely, there arises a natural question whether closed forms of degree k are always exact, and we will find later that in the case of $\mathbb{R}^n$, this is true for $k > 0$ (§3.3, Poincaré lemma (Corollary 3.14)). However, in the case of general C^∞ manifolds, a closed form is not always exact, and the "gap" will reflect the global structure of manifolds. This is the content of the theory of de Rham cohomology, which is the theme of Chapter 3.

Since the proof of the following proposition is easy, we leave it to the reader (Exercise 2.1).

PROPOSITION 2.3. *For $\omega \in \mathcal{A}^k(\mathbb{R}^n)$ and $\eta \in \mathcal{A}^l(\mathbb{R}^n)$, we have*

(i) $\eta \wedge \omega = (-1)^{kl}\omega \wedge \eta$,
(ii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

Now let U, U' be two open sets in $\mathbb{R}^n$ and $\varphi : U \to U'$ a diffeomorphism. Then a homomorphism

$$\varphi^* : \mathcal{A}^*(U') \longrightarrow \mathcal{A}^*(U)$$

from the algebra $\mathcal{A}^*(U')$ of all differential forms on U' to the algebra $\mathcal{A}^*(U)$ of all differential forms on U is defined as follows. For an arbitrary function $f \in \mathcal{A}^0(U')$, let $\varphi^*(f) = f \circ \varphi \in \mathcal{A}^0(U)$ and let $\varphi^*(dx_i) = d(\varphi^*(x_i))$. We extend this to differential forms of general degree in such a way that

$$\varphi^*(\omega \wedge \eta) = \varphi^*(\omega) \wedge \varphi^*(\eta)$$

for an exterior product $\omega \wedge \eta$ of two differential forms. Practically, we proceed as follows. Let the coordinates of U' be $x_1, \cdots, x_n$ and the coordinates of U be $y_1, \cdots, y_n$ (to distinguish these from the coordinates of U'). Then each x_i is written as a function $x_i = x_i(y_1, \cdots, y_n)$ of $y_1, \cdots, y_n$. Then we have $\varphi^*(dx_i) = \sum_j \frac{\partial x_i}{\partial y_j} dy_j$, and from this we see that

(2.4)

$$\varphi^*(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j_1 < \cdots < j_k} \frac{D(x_{i_1}, \cdots, x_{i_k})}{D(y_{j_1}, \cdots, y_{j_k})} dy_{j_1} \wedge \cdots \wedge dy_{j_k}.$$

Here $\dfrac{D(x_{i_1},\cdots,x_{i_k})}{D(y_{j_1},\cdots,y_{j_k})}$ denotes the Jacobian of $x_{i_1},\cdots,x_{i_k}$ with respect to $y_{j_1},\cdots,y_{j_k}$. Then we see (verification is Exercise 2.2) that

$$d\circ\varphi^* = \varphi^*\circ d,$$

and by the consideration of φ^{-1}, we can verify that φ^* is in fact an isomorphism.

Henceforth, $\varphi^*(\omega)$ will sometimes be denoted simply by $\varphi^*\omega$.

(b) Differential forms on a general manifold.

Let M be an n-dimensional C^∞ manifold and $\{(U_\alpha,\varphi_\alpha)\}$ an atlas of it. In brief, a degree k differential form on M is a family $\{\omega_\alpha\}$ of k-forms ω_α on each coordinate neighborhood U_α (which can be considered as an open set of $\mathbb{R}^n$) such that for arbitrary α,β with $U_\alpha\cap U_\beta\neq\varnothing$, ω_α and ω_β are transformed to each other in the sense of (2.4) by the coordinate change. We denote the set of all k-forms on M by $\mathcal{A}^k(M)$, and we put

$$\mathcal{A}^*(M)=\bigoplus_{k=0}^{n}\mathcal{A}^k(M).$$

As we saw in the previous subsection (a), the homomorphism $\varphi^*:\mathcal{A}^*(U')\to\mathcal{A}^*(U)$ between algebras of all differential forms induced by a coordinate change preserves the exterior products and commutes with the operation of exterior differentiation. From this, we see that the exterior products and the exterior differentiation $d:\mathcal{A}^k(M)\to\mathcal{A}^{k+1}(M)$ are defined also on $\mathcal{A}^*(M)$, and $d\circ d=0$. Furthermore, Proposition 2.3 holds for differential forms on M.

Although this definition is right, the formula (2.4) is fairly complicated, and from the standpoint of studying the whole M it may not give a good insight. Therefore, we shall define these differential forms independently of the local coordinates. We need to prepare some abstract facts for it. It is not appropriate to ask which of these two definitions is better, and the important thing is that we learn from them what differential forms are after all.

(c) The exterior algebra.

We shall start by giving the relationship between the exterior algebra Λ_n^* generated by $dx_1,\cdots,dx_n$ and the tangent space $T_0\mathbb{R}^n$ of $\mathbb{R}^n$ at the origin. $T_0\mathbb{R}^n$ is an n-dimensional vector space with a basis $\dfrac{\partial}{\partial x_1},\ldots,\dfrac{\partial}{\partial x_n}$. On the other hand, each dx_i can be considered as an element of the dual space

$$T_0^*\mathbb{R}^n = \{\alpha : T_0\mathbb{R}^n \to \mathbb{R}\ ;\ \alpha \text{ a linear map}\}$$

of $T_0\mathbb{R}^n$. This is because x_i can be considered as a C^∞ function $x_i : \mathbb{R}^n \to \mathbb{R}$ and the differential $dx_i : T_0\mathbb{R}^n \to T_0\mathbb{R} = \mathbb{R}$ of this function at the origin is linear. Then obviously

$$dx_i\Big(\frac{\partial}{\partial x_j}\Big) = \delta_{ij}. \tag{2.5}$$

From another point of view, since $\dfrac{\partial}{\partial x_j}$ is a unit tangent vector in the direction of x_j, we can consider that (2.5) reflects the fact that if we integrate the constant function 1 with respect to x_i from 0 to 1 along the x_j-axis, the value is δ_{ij}. Thus Λ_n^1 is identified with $T_0^*\mathbb{R}^n$:

$$\Lambda_n^1 = T_0^*\mathbb{R}^n.$$

In general, an arbitrary element in Λ_n^k is described as a linear combination of the elements of the form $\omega = \alpha_1 \wedge \cdots \wedge \alpha_k$ $(\alpha_i \in \Lambda_n^1)$, while such an ω defines a map

$$\omega : \underbrace{T_0\mathbb{R}^n \times \cdots \times T_0\mathbb{R}^n}_{k} \to \mathbb{R} \tag{2.6}$$

as follows. That is, for $X_i \in T_0\mathbb{R}^n$ $(i = 1, \cdots, k)$, we put

$$\omega(X_1, \cdots, X_k) = \frac{1}{k!}\det\Big(\alpha_i(X_j)\Big). \tag{2.7}$$

Here, $\Big(\alpha_i(X_j)\Big)$ denotes a matrix whose (i,j)-entry is $\alpha_i(X_j)$. Using the properties of determinant, it is easy to see that the above value is uniquely determined, independently of the expression of ω. For example, if we write $\omega = -\alpha_2 \wedge \alpha_1 \wedge \alpha_3 \wedge \cdots \wedge \alpha_k$, the value is the same. The geometric meaning of this value is roughly as follows. For example, $dx_1 \wedge dx_2(X_1, X_2)$ is the (signed) area of the orthogonal projection of the triangle spanned by two tangent vectors X_1, X_2 in $T_0\mathbb{R}^n$ onto the (x_1, x_2)-direction, and in general, (2.7) is considered to present "the (signed) volume in the direction of $(\alpha_1, \cdots, \alpha_k)$" of the k-dimensional simplex (a generalization of triangle, see §3.1) spanned by $X_1, \cdots, X_k$. If we recall these facts when we define the integration of differential forms on manifolds later in Chapter 3, it may help our understanding. For a general element $\omega \in \Lambda_n^k$, the map (2.6) is also defined by extending the above definition linearly.

We see that the map ω of (2.6) has the following two properties. Since the proof can be given easily by using the properties of determinant, we leave it to the reader.

(i) ω is **multilinear**. That is, for an arbitrary X_i, it satisfies the linearity condition

$$\omega(X_1, \cdots, X_{i-1}, aX_i + bX_i', X_{i+1}, \cdots, X_k)$$
$$= a\omega(X_1, \cdots, X_i, \cdots, X_k) + b\omega(X_1, \cdots, X_i', \cdots, X_k).$$

(ii) ω is **alternating**. That is, for arbitrary $i < j$, if we interchange X_i and X_j, its sign changes. Therefore for an arbitrary permutation $\sigma \in \mathfrak{S}_n$ of n letters,

$$\omega(X_{\sigma(1)}, \cdots, X_{\sigma(n)}) = \operatorname{sgn}\sigma\, \omega(\mathrm{X}_1, \cdots, \mathrm{X}_n).$$

Here $\operatorname{sgn}\sigma$ denotes the sign of σ.

We call the map $T_0\mathbb{R}^n \times \cdots \times T_0\mathbb{R}^n$ (n-fold direct product) $\to \mathbb{R}$ satisfying the above two conditions an **alternating form** of degree k on $T_0\mathbb{R}^n$. As a result, by the correspondence (2.6), a map

(2.8) $$\Lambda_n^k \cong \text{all alternating forms of degree } k \text{ on } \mathrm{T}_0\mathbb{R}^n$$

is defined, and this turns out to be a one to one correspondence. Here, the right-hand side of (2.8) does not contain the coordinates x_i of $\mathbb{R}^n$ and is presented purely in terms of linear algebra. With this in mind as a clue to go on, we shall give a definition of differential forms on general manifolds which is **independent of the coordinates**. We shall describe it without worrying about some repetition.

Let V be a vector space over $\mathbb{R}$. Since we need only the case of tangent space T_0M at a point p on a C^∞ manifold M, it may be read as $V = T_pM$. The **dual space** V^* of V is a vector space defined as

$$V^* = \{\alpha : V \to \mathbb{R};\ \alpha \text{ a linear map }\}.$$

DEFINITION 2.4. Let V be a vector space over $\mathbb{R}$. An algebra with unit 1 generated by the elements of V over $\mathbb{R}$ satisfying the relation

(2.9) $$X \wedge Y = -Y \wedge X$$

for arbitrary $X, Y \in V$ is denoted by Λ^*V and called an **exterior algebra** of V or a **Grassmann algebra**. Here $\wedge$ stands for the product of this algebra.

By condition (2.9), $X \wedge X = 0$ for an arbitrary $X \in V$. Conversely, it is easy to see that (2.9) follows from this condition. The previous Λ_n^* is nothing but $\Lambda^* T_0^* \mathbb{R}^n$. In the same way as in the case of Λ_n^*, if $\dim V = n$, we have a direct sum decomposition

$$\Lambda^* V = \bigoplus_{k=0}^{n} \Lambda^k V.$$

Here $\Lambda^k V$ is the subspace of $\Lambda^* V$ consisting of all elements of degree k. Let $e_1, \cdots, e_n$ be a basis of V. Then we can take

$$e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n \tag{2.10}$$

as a basis of $\Lambda^k V$, and therefore $\dim \Lambda^k V = \binom{n}{k}$. Also, $\Lambda^0 V = \mathbb{R}$ and $\Lambda^1 V$ can be naturally identified with V. While we defined the exterior algebra of V, the exterior algebra $\Lambda^* V^*$ of V^* is also defined similarly. It is this case that we use later.

Next we shall define alternating forms on V.

DEFINITION 2.5. Let V be a vector space over $\mathbb{R}$. A multilinear map

$$\omega : \underbrace{V \times \cdots \times V}_{k} \longrightarrow \mathbb{R}$$

from k-fold direct product of V to $\mathbb{R}$ that is alternating, namely

$$\omega(X_{\sigma(1)} \cdots X_{\sigma(k)}) = \operatorname{sgn} \sigma \omega(X_1, \cdots, X_k) \quad (X_i \in V)$$

for an arbitrary permutation σ of k letters, is called an alternating form of degree k on V.

The set of all alternating forms of degree k on V is denoted by $\mathcal{A}^k(V)$. $\mathcal{A}^k(V)$ is a vector space with respect to the natural sum and the multiplication of alternating forms by real numbers. We shall consider all alternating forms

$$\mathcal{A}^*(V) = \bigoplus_{k=0}^{\infty} \mathcal{A}^k(V)$$

with different degrees on V. Here we define $\mathcal{A}^0(V) = \mathbb{R}$, and it is easy to see that $\mathcal{A}^k(V) = 0$ for $k > \dim V$, by the alternating condition.

A degree preserving linear map

$$\iota : \Lambda^* V^* \longrightarrow \mathcal{A}^*(V)$$

from the exterior algebra $\Lambda^* V^*$ of the dual space V^* of V to the vector space $\mathcal{A}^*(V)$ of all alternating forms on V is defined as follows. It is enough to define

$$\iota_k : \Lambda^k V^* \longrightarrow \mathcal{A}^k(V)$$

for each k. For an element of the form $\omega = \alpha_1 \wedge \cdots \wedge \alpha_k \in \Lambda^k V^*$ ($\alpha_i \in V^*$), we set

$$\iota_k(\omega)(X_1, \cdots, X_k) = \frac{1}{k!}\det\Big(\alpha_i(X_j)\Big)$$

and extend it linearly for general elements. It is easy to see that ι_k is well defined independently of the expression of ω, by using the properties of determinant in the same way as before.

PROPOSITION 2.6. *The map $\iota : \Lambda^* V^* \longrightarrow \mathcal{A}^*(V)$ is an isomorphism. That is, the exterior algebra $\Lambda^* V^*$ of V^* and the vector space $\mathcal{A}^*(V)$ of all alternating forms on V can be identified by ι. Using this, a product is defined on $\mathcal{A}^*(V)$ which is described as follows. If for $\omega \in \Lambda^k V^*, \eta \in \Lambda^l V^*$, we consider their exterior product $\omega \wedge \eta$ as an element of $\mathcal{A}^{k+l}(V)$ by the identification ι, we have*

(2.11)

$$\begin{aligned} &\omega \wedge \eta\ (X_1, \cdots, X_{k+l}) \\ &= \frac{1}{(k+l)!}\sum_{\sigma} \operatorname{sgn}\sigma\ \omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)})\ \eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}) \\ &\qquad\qquad (X_i \in V). \end{aligned}$$

Here σ runs over the set $\mathfrak{S}_{k+l}$ of all permutations of $k+l$ letters $1, 2, \cdots, k+l$.

PROOF. First, we show that ι_k is an isomorphism. Let $e_1, \cdots, e_n$ be a basis of V and $\alpha_1, \cdots, \alpha_n$ its dual basis of V^*. They satisfy $\alpha_i(e_j) = \delta_{ij}$. Then by (2.10) we can take

$$\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n$$

as a basis of $\Lambda^k V^*$. We can check that the images of elements of this basis by ι are linearly independent as elements of $\mathcal{A}^k(V)$ by applying them to

$$(e_{j_1}, \cdots, e_{j_k}) \in V \times \cdots \times V, \quad j_1 < \cdots < j_k.$$

Next, let $\omega \in \mathcal{A}^k(V)$ be an arbitrary element. Then if we set $\omega(e_{i_1}, \cdots, e_{i_k}) = a_{i_1 \cdots i_k}$ and, using these constants, define

$$\widetilde{\omega} = k! \sum_{i_1 \cdots i_k} a_{i_1 \cdots i_k} \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \in \Lambda^k V^*,$$

we see that $\iota_k(\widetilde{\omega}) = \omega$. Therefore, ι_k is a surjection and hence an isomorphism.

Next, we prove the latter half of the claim. It is enough to prove it for the elements ω, η of the form

$$\omega = \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}, \quad \eta = \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_l}$$

by the linearity of ι_k. Furthermore, we may assume that $i_1, \cdots, i_k$, $j_1, \cdots, j_l$ are all distinct, because otherwise we have $\omega \wedge \eta = 0$. Then we rearrange these numbers in order of size so that $m_1 < \cdots < m_{k+l}$. If we let the permutation of rearrangement be τ, we have

$$\omega \wedge \eta = \operatorname{sgn} \tau \alpha_{m_1} \wedge \cdots \wedge \alpha_{m_{k+l}}.$$

Therefore,

$$\iota_{k+l}(\omega \wedge \eta)(e_{m_1}, \cdots, e_{m_{k+l}}) = \frac{1}{(k+l)!} \operatorname{sgn} \tau.$$

On the other hand, if we calculate

$$\sum_{\sigma} \operatorname{sgn} \sigma \ \iota_k(\omega)(e_{m_{\sigma(1)}}, \cdots, e_{m_{\sigma(k)}}) \ \iota_l(\eta)(e_{m_{\sigma(k+1)}}, \cdots, e_{m_{\sigma(k+l)}}),$$

we see that it is $\operatorname{sgn} \tau$. In this way, we see that the claim is true for $(e_{m_1}, \cdots, e_{m_{k+l}})$. But since for every other element of the form $(e_{n_1}, \cdots, e_{n_{k+l}})$ the value is 0, the proof finishes. ∎

The above isomorphism $\iota : \Lambda^* V^* \cong \mathcal{A}^*(V)$ is not the unique natural one. Actually, if we let $\iota'_k = k! \iota_k$, we obtain another isomorphism $\iota' : \Lambda^* V^* \cong \mathcal{A}^*(V)$, and this defines another product on $\mathcal{A}^*(V)$ (however, the difference between the two products is only up to scalars and is not essential). This is equivalent to considering the volume of the parallelotope spanned by each vector instead of the volume of the k-dimensional simplex defined by the origin and the end point of each vector in the description following (2.7). While these two methods have their own merits, we use ι in this book because there are some inconveniences with ι' when we describe the general theory of characteristic classes in Chapter 6. However, since ι' is defined over $\mathbb{Z}$, it has the advantage of eliminating fractional constants in various formulae. For example, the coefficient $\frac{1}{k+1}$ in the formula of exterior differentiation (Theorem 2.9) is not necessary if we use ι'.

(d) Various definitions of differential forms.

While we have already defined differential forms on general C^∞ manifolds in subsection (b), in this subsection we shall give a more intrinsic definition without using local coordinates.

The dual space T_p^*M of the tangent space T_pM at a point p on M is called the **cotangent space** at p. By the description in the previous subsection, we can consider its exterior algebra $\Lambda^* T_p^* M$.

DEFINITION 2.7. Let M be a C^∞ manifold. We say that ω is a k-form on M if it assigns $\omega_p \in \Lambda^* T_p^* M$ to each point $p \in M$ and ω_p is of class C^∞ with respect to p.

Let U be an arbitrary coordinate neighborhood, and $x_1, \cdots, x_n$ coordinate functions defined on U. Then, for any point $p \in U$,

$$\left(\frac{\partial}{\partial x_1}\right)_p, \cdots, \left(\frac{\partial}{\partial x_n}\right)_p$$

become a basis of the tangent space T_pM. We shall find the dual basis for the dual space T_p^*M. Each x_i can be regarded as a C^∞ function $x_i : U \to \mathbb{R}$. Consider the differential $(dx_i)_p : T_pM \to T_{x_i(p)}\mathbb{R}$ of this map at p. Since $T_{x_i(p)}\mathbb{R}$ can be naturally identified with $\mathbb{R}$, we can consider $(dx_i)_p$ as an element in T_p^*M. Then obviously,

$$dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$$

(see (2.5)). Therefore,

$$(dx_1)_p, \cdots, (dx_n)_p$$

become the dual basis of T_p^*M. It follows from this fact that ω_p in the above Definition 2.7 is presented as

$$\omega_p = \sum_{i_1 < \cdots < i_k} f_{i_1 \cdots i_k}(p)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{2.12}$$

ω_p is said to be of class C^∞ if each coefficient $f_{i_1 \cdots i_k}(p)$ is of class C^∞ as a function of p. The expression (2.12) is called the local expression of the k-form ω on M. Thus, Definition 2.7 and the definition in subsection (b) are related.

If we use the terminology of vector bundles which will appear in Chapter 5, we can interpret the above as follows. If we set

$$T^*M = \bigcup_p T_p^*M,$$

it is easy to see that this is a vector bundle over M. We call this the **cotangent bundle** of M. Similarly, if we set

$$\Lambda^k T^*M = \bigcup_p \Lambda^k T_p^* M,$$

this is also a vector bundle over M. Note that $\Lambda^1 T^*M = T^*M$. In these terms, k-forms on M are nothing but sections of $\Lambda^k T^*M$ of class C^∞. That is,

$$\mathcal{A}^k(M) = \text{all sections of } \Lambda^k T^*M \text{ of class } C^\infty.$$

Finally, we mention another view of differential forms. Let ω be a k-form on M. Then the value ω_p of ω at each point p determines an alternating form $T_pM \times \cdots \times T_pM \to \mathbb{R}$ of degree k. Putting all p together, ω induces a multi-linear and alternating map

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \longrightarrow C^\infty(M). \tag{2.13}$$

Here $\mathfrak{X}(M)$ denotes the set of all vector fields on M and $C^\infty(M)$ denotes the algebra of all C^∞ functions on M. It is important here that $\mathfrak{X}(M)$ is not only a vector space over $\mathbb{R}$ but also a module over $C^\infty(M)$. That is, for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, fX is also a vector field on M. Then the meaning of (2.13) being multilinear is that it is also linear with respect to the multiplication of vector fields by functions. More precisely,

$$\begin{aligned}&\omega(X_1, \cdots, fX_i + gX_i', \cdots, X_k)\\ &\quad= f\omega(X_1, \cdots, X_i, \cdots, X_k) + g\omega(X_1, \cdots, X_i', \cdots, X_k)\end{aligned}$$

for arbitrary $X_i \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Conversely, we see that any map (2.13) with these two properties (that is, multilinear as a $C^\infty(M)$ module and alternating) defines a differential form. Namely, the following theorem holds.

THEOREM 2.8. *Let M be a C^∞ manifold. Then the set $\mathcal{A}^k(M)$ of all k-forms on M can be naturally identified with that of all multilinear and alternating maps, as $C^\infty(M)$ modules, from k-fold direct product of $\mathfrak{X}(M)$ to $C^\infty(M)$.*

PROOF. Suppose that a map $\tilde{\omega} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$ with the above conditions is given. First of all, we shall see that for arbitrary vector fields $X_i \in \mathfrak{X}$, the value $\tilde{\omega}(X_1, \cdots, X_k)(p)$ at a point p is determined depending only on the values $X_i(p)$ of each vector field X_i at p. For that, by linearity, it is enough to show that if $X_i(p) = 0$ for some i, then the above value is 0. For the sake of simplicity, assume that $i = 1$, and let $(U; x_1, \cdots, x_n)$ be a local coordinate system around p. Then we can write $X_1 = \sum_i f_i \dfrac{\partial}{\partial x_i}$ on U with $f_i(p) = 0$. We choose an open neighborhood V of p such that $\overline{V} \subset U$, and a C^∞ function $h \in C^\infty(M)$ such that it is identically 1 on V and 0 outside of U(see Lemma 1.28). Let $Y_i = h\,\dfrac{\partial}{\partial x_i}$. Then we have $Y_i \in \mathfrak{X}(M)$, and if we set $\widetilde{f}_i = h\, f_i$, then we have $\widetilde{f}_i \in C^\infty(M)$. Now it is easy to see that

$$X_1 = \sum_i \widetilde{f}_i\, Y_i + (1 - h^2) X_1.$$

Therefore, we have

$$\begin{aligned}&\widetilde{\omega}(X_1, \cdots, X_k)(p)\\ &= \sum_i \widetilde{f}_i(p)\, \widetilde{\omega}(Y_i, X_2, \cdots, X_k)(p) + (1 - h(p)^2)\widetilde{\omega}(X_1, \cdots, X_k)(p) = 0,\end{aligned}$$

and the claim is proved.

Now we define a k-form ω as follows. At each point $p \in M$, if tangent vectors $X_1, \cdots, X_k \in T_pM$ are given, we choose vector fields $\widetilde{X}_i$ over M such that $\widetilde{X}_i(p) = X_i$. If we let $\omega_p(X_1, \cdots, X_k) = \widetilde{\omega}(\widetilde{X}_1, \cdots, \widetilde{X}_k)(p)$, then, as we saw above, this is determined independently of the choice of $\widetilde{X}_i$. Since it is easy to see that ω_p is of class C^∞ with respect to p, ω is the required differential form. ∎

2.2. Various operations on differential forms

Let M be an n-dimensional C^∞ manifold. We denote all k-forms on M by $\mathcal{A}^k(M)$ and consider their direct sum

$$\mathcal{A}^*(M) = \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$$

with respect to k, that is, the set of all differential forms on M. In this section, we shall define various operations on $\mathcal{A}^*(M)$.

(a) Exterior product.

The **exterior product** $\omega \wedge \eta \in \mathcal{A}^{k+l}(M)$ of a k-form $\omega \in \mathcal{A}^k(M)$ and an l-form $\eta \in \mathcal{A}^l(M)$ on M is defined as follows. Since at each point $p \in M$ we have $\omega_p \in \Lambda^k T_p^* M$, $\eta_p \in \Lambda^l T_p^* M$, their product $\omega_p \wedge \eta_p \in \Lambda^{k+l} T_p^* M$ is defined. Then, we put

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p.$$

By definition, the exterior product is obviously associative. That is, if $\tau \in \mathcal{A}^m(M)$, we have $(\omega \wedge \eta) \wedge \tau = \omega \wedge (\eta \wedge \tau)$. Therefore we do not need the parentheses. If they are locally expressed as $\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \eta = g\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$, we have

$$\omega \wedge \eta = fg\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

The exterior product induces a bilinear map

$$\mathcal{A}^k(M) \times \mathcal{A}^l(M) \ni (\omega, \eta) \mapsto \omega \wedge \eta \in \mathcal{A}^{k+l}(M)$$

and it has the following properties.

(i) $\eta \wedge \omega = (-1)^{kl}\, \omega \wedge \eta$.

(ii) For arbitrary vector fields $X_1, \cdots, X_{k+l} \in \mathfrak{X}(M)$,

(2.14)

$$\omega \wedge \eta(X_1, \cdots, X_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \operatorname{sgn} \sigma \omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}).$$

Property (i) is obvious from the description above, and (ii) follows from (2.11).

(b) Exterior differentiation.

For a k-form $\omega \in \mathcal{A}^k(M)$ on M, its **exterior differentiation** $d\omega \in \mathcal{A}^{k+1}(M)$ is the operation defined by

$$d\omega = \sum_j \frac{\partial f}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k};$$

here ω is locally expressed as $\omega = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. In view of the fact that for the isomorphism $\varphi^* : \mathcal{A}^*(U') \to \mathcal{A}^*(U)$ induced by an arbitrary diffeomorphism $\varphi : U \to U'$ between two open sets U, U' of $\mathbf{R}^n$, the equation $d \circ \varphi^* = \varphi^* \circ d$ holds (see the description following (2.4)), we see that the above d does not depend on the local expression. Therefore, the operation of taking the exterior differentiation defines a degree 1 (that is, increasing the degree by 1) linear map

$$d : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{k+1}(M),$$

and from Lemma 2.2 and Proposition 2.3, we see that it has the following properties.

(i) $d \circ d = 0$.

(ii) For $\omega \in \mathcal{A}^k(M)$, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k\, \omega \wedge d\eta$.

Next, we shall characterize the exterior differentiation without using the local expression. Namely, we have the following theorem.

THEOREM 2.9. *Let M be a C^∞ manifold and $\omega \in \mathcal{A}^k(M)$ an arbitrary k-form on M. Then for arbitrary vector fields $X_1, \cdots, X_{k+1} \in \mathfrak{X}(M)$, we have*

$$d\omega(X_1, \cdots, X_{k+1}) = \frac{1}{k+1}\Big\{\sum_{i=1}^{k+1} (-1)^{i+1} X_i(\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1})) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_{k+1})\Big\}.$$

Here the symbol $\widehat{X}_i$ means X_i is omitted. In particular, the often-used case of $k = 1$ is

$$d\omega(X, Y) = \frac{1}{2}\{X\omega(Y) - Y\omega(X) - \omega([X, Y])\} \quad (\omega \in \mathcal{A}^1(M)).$$

PROOF. If we consider the right-hand side of the formula to be proved, as a map from the $(k+1)$-fold direct product of $\mathfrak{X}(M)$ to $C^\infty(M)$, we see that it satisfies the conditions of degree $k+1$ alternating form as a map between modules over $C^\infty(M)$. Since it is easy to verify this fact by using Proposition 1.40 (iv), we leave it to the reader. Therefore, by Theorem 2.8, we see that the right-hand side is a $(k+1)$-form on M.

If two differential forms coincide in some neighborhood of an arbitrary point, they coincide on the whole. Then, consider a local coordinate system $(U; x_1, \cdots .x_n)$ around an arbitrary point $p \in M$. Let the local expression of ω with respect to this local coordinate system be $\omega = \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then, we have

$$d\omega = \sum_{i_1<\cdots<i_k} df_{i_1\cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}. \tag{2.15}$$

From the linearity of differential forms with respect to the functions on M, it is enough to consider only vector fields X_i such that $X_i = \dfrac{\partial}{\partial x_{j_i}}$ $(i = 1, \cdots, k+1)$ in a neighborhood of p. Then $[X_i, X_j] = 0$ near p. Moreover, by the alternating property of differential forms, we may assume that $j_1 < \cdots < j_{k+1}$. Then, if we apply (2.15) to

$(X_1, \cdots, X_{k+1})$, we have

$$d\omega(X_1, \cdots, X_{k+1}) = \frac{1}{(k+1)!}\left\{\sum_{s=1}^{k+1} (-1)^{s-1} \frac{\partial}{\partial x_{j_s}} f_{j_1 \cdots \hat{j}_s \cdots j_{k+1}}\right\}.$$

On the other hand, when we calculate the right hand side of the formula using $[X_i, X_j] = 0$, we obtain the same value. This finishes the proof. ■

We can consider Theorem 2.9 as a definition of the exterior differentiation that is independent of the local coordinates.

(c) Pullback by a map.

We shall study the relationship between differential forms and C^∞ maps. Let

$$f : M \longrightarrow N$$

be a C^∞ map from a C^∞ manifold M to N. Consider the differential $f_* : T_pM \to T_{f(p)}N$ of f at each point $p \in M$. f_* induces its dual map $f^* : T^*_{f(p)}N \to T^*_pM$, that is, the map defined by $f^*(\alpha)(X) = \alpha(f_*(X))$ for $\alpha \in T^*_{f(p)}N$, $X \in T_pM$. Furthermore, f^* defines a linear map $f^* : \Lambda^k T^*_{f(p)}N \to \Lambda^k T^*_p M$ for an arbitrary k, and they induce an algebra homomorphism

$$f^* : \mathcal{A}^*(N) \longrightarrow \mathcal{A}^*(M).$$

For a differential form $\omega \in \mathcal{A}^k(N)$ on N, $f^*\omega \in \mathcal{A}^k(M)$ is called the **pullback** by f. Explicitly, for $X_1, \cdots, X_k \in T_pM$,

$$f^*\omega(X_1, \cdots, X_k) = \omega(f_*X_1, \cdots, f_*X_k).$$

PROPOSITION 2.10. *Let M, N be C^∞ manifolds. Let $f : M \to N$ be a C^∞ map and $f^* : \mathcal{A}^*(N) \to \mathcal{A}^*(M)$ the map induced by f. Then f^* is linear and has the following properties.*

(i) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta \quad (\omega \in \mathcal{A}^k(N),\ \eta \in \mathcal{A}^l(N))$.

(ii) $d(f^*\omega) = f^*(d\omega) \quad (\omega \in \mathcal{A}^k(M))$.

Since the proof can be given easily by using the previous results, we leave it to the reader.

(d) Interior product and Lie derivative.

Let M be a C^∞ manifold and $X \in \mathfrak{X}(M)$ a vector field on M. Then a linear map

$$i(X) : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{k-1}(M)$$