Note that if u is u.s.c., it is bounded above on every compact $K \subset \Omega$; in fact K is contained in the union of the increasing sequence of open sets $\{z \in \Omega \mid u(z) < \nu\}$, so in one of them.

Lemma 1. Let u be u.s.c. in Ω and bounded above. Then there is a sequence $\{u_k\}$ of continuous functions on Ω such that $u_k(z)$ decreases to $u(z)$ for all $z \in \Omega$.

Proof. We set $u_k(z) = \sup_{\zeta \in \Omega} \{u(\zeta) - k|\zeta - z|\}$, $M = \sup_{\zeta \in \Omega} u(\zeta)$. Clearly $-\infty < u_k(z) < +\infty$ for all $z \in \Omega$. We have $u_k(z) \geq u(z) - k|z-z| = u(z)$. Further, since the sequence $\{u(\zeta) - k|\zeta - z|\}$ decreases as k increases (for fixed ζ, z), $u_k(z)$ decreases for every z. Further, if $z, z' \in \Omega$,

$$u_k(z) \geq u(\zeta) - k|\zeta - z| \geq u(\zeta) - k|\zeta - z'| - k|z - z'|, \quad \forall \zeta \in \Omega,$$

so that $u_k(z) \geq u_k(z') - k|z - z'|$. Hence $|u_k(z) - u_k(z')| \leq k|z - z'|$, so that u_k is continuous on Ω.

To prove that $u_k(z) \to u(z)$ as $k \to \infty$, suppose first that $u(z) > -\infty$. Let $\varepsilon > 0$ and $\Omega' = \{z' \in \Omega \mid u(z') < u(z) + \varepsilon\}$; Ω' is an open neighborhood of z, and contains a disc $|z' - z| < \delta$. Let k_o be such that $M - k_o\delta < u(z)$. Then $u(z') - k|z' - z| \leq u(z') < u(z) + \varepsilon$, for $z' \in \Omega'$, while $u(z') - k|z' - z| < M - k_o\delta < u(z)$ for $z' \notin \Omega'$, $k \geq k_o$. Hence, for $k \geq k_o$,

$$u(z) \leq u_k(z) < u(z) + \varepsilon, \text{ so that } u_k(z) \to u(z) \text{ as } k \to \infty.$$

If $u(z) = -\infty$, and $c > 0$, then

$$\Omega' = \{z' \in \Omega \mid u(z) < -c\}$$

contains a disc $|z' - z| < \delta$, so that

$$u_k(z) \leq \max(-c, M - k\delta)$$

as before, and $u_k(z) \to -\infty$ as $k \to \infty$.