

2024 spring. Quantum graphs

Origin of quantum graphs

1. paradigm of quantum sets

operator alg. $\equiv$ func. space on quantized set
 $M \subset \mathcal{B}(H)$

Weaver 2012: operator system $V \subset \mathcal{B}(H)$

s.t. $V = M' \cdot V \cdot M'$ should be
 regarded as quant. graph.

$$M' = \{ S \in \mathcal{B}(H) : \forall T \in M \quad ST = TS \}$$

$$\text{op. sys} \quad S \in V \Rightarrow S^* \in V, \text{Id}_H \in V$$

2. confusability graph of quantum channels

classically: channel F with input set I
 output set O

$\leadsto$ graph Γ with vert. set I
 (unoriented)

$$x \sim y \quad \text{if} \quad P[F(x) = F(y)] \neq 0 \text{ prob.}$$

$F(x)$ rand. var. on O for each $x \in I$.

Duan - Severini - Winter 2013, Stahlke 2016

confusability graph for quantum channel

$$N: \mathcal{B}(H_I) \rightarrow \mathcal{B}(H_O), \quad p \mapsto \sum_i K_i p K_i^* \quad \text{CP, tr-pres.}$$

$\leadsto V = \text{span}_{i,j} \langle K_i^* K_j \rangle$ represents the confus.
 graph of N .

What do we compute for quantum graphs?

1. algebras representing "functions" on the sp. of "homomorphisms" / "isomorphisms" of quantum graphs

→ connects with quantum groups (Hopf algs)
nonlocal games
nontriv. already for usual graphs

2. numerical invariants inspired by graph theory

Lovász number $\vartheta(\Gamma) = \min_{A=[a_{ij}]} \lambda_{\max}(A)$ $\begin{matrix} a_{ii} = 1 \\ i \neq j \Rightarrow a_{ij} = 0 \end{matrix}$ (

chromatic number $\chi(\Gamma) = \min \{ k : \exists \text{ coloring by } k\text{-colors } s, t, v \sim w \Rightarrow c_s \neq c_w \}$

→ analogue for quantum graphs?

3. estimate for quantum channels

zero-error capacity, etc.

First definitions

we follow Daws Section 2

finite dim C^* -alg. $B \cong \bigoplus_{k=1}^n M_{d_k}$

faithful state : functional $\psi : B \rightarrow \mathbb{C}$

- positive $\psi(a^*a) \geq 0$

- normalized $\psi(1_B) = 1$

- faithful $\psi(a^*a) = 0 \Leftrightarrow a = 0$

Gelfand - Naimark - Segal space. $L^2(B, \psi)$

underlying vec. sp. $\{ \Lambda(a) : a \in B \}$

inner prod. $(\Lambda(a) | \Lambda(b)) = \psi(a^*b)$

"unit map" $\eta: \mathbb{C} \rightarrow L^2(B, \psi)$, $\lambda \mapsto \lambda \wedge (1_B)$

"prod. map" $m: L^2(B, \psi)^{\otimes 2} \rightarrow L^2(B, \psi)$, $\wedge(a) \otimes \wedge(b) \mapsto \wedge(ab)$

$\leadsto$ their adjoints $\eta^*: L^2(B, \psi) \rightarrow \mathbb{C}$
 $m^*: L^2(B, \psi) \rightarrow L^2(B, \psi)^{\otimes 2}$

Exercise: check $\eta^*(\wedge(a)) = \psi(a)$

Def. a quantum adjacency matrix on (B, ψ) is an operator $A \in \mathcal{B}(L^2(B, \psi))$ s.t.

0. $A = A^*$
1. $m(A \otimes A) m^* = A$
2. $(\text{id} \otimes \eta^* m)(\text{id} \otimes A \otimes \text{id})(m^* \eta \otimes \text{id}) = A$
3. $m(A \otimes \text{id}) m^* = \text{id}$

a quantum graph is given by B, ψ, A as above

graphical calculus

$$\begin{array}{llll} \eta: & \text{point}^B & m: & \text{cup}^B \\ A: & \text{box } A^B & A^*: & \text{box } A^B \\ & & & \text{box } A^B = \text{box } A^B \end{array}$$

$$\leadsto \text{box } A \text{ box } A \stackrel{(1.)}{=} \text{box } A, \quad \text{box } A \stackrel{(2.)}{=} \text{box } A, \quad \text{box } A \stackrel{(3.)}{=} \text{box } A$$

standard choice for ψ :

$$B = \bigoplus_{k=1}^n M_{d_k} \leadsto \psi = \left(\bigoplus_k d_k \text{Tr} \right) / \left(\sum_k d_k^2 \right)$$

$$\text{we get } m m^* = \left(\sum d_k^2 \right) \text{id}_B. \quad (\delta\text{-form property})$$