

Classical graphs $\subset$ quantum graphs

Recall: a quant. graph was given by

- finite dim. C^* -alg $B = \bigoplus_{k=1}^n M_{d_k}$
- faithful state func. $\psi: B \rightarrow \mathbb{C}$
- quant. adjacency matrix op. $A \in B(L^2(B, \psi))$

$$A = A^*, \quad \begin{array}{c} \text{loop } m \\ \boxed{A} \end{array} = \begin{array}{c} \text{loop } m^* \\ \boxed{A} \end{array} = \begin{array}{c} \text{loop } \eta^* m \\ \boxed{A} \end{array}, \quad \begin{array}{c} \text{loop } 1 \\ \boxed{A} \end{array} = | \text{id}_{L^2(B, \psi)} |$$

How to interpret usual graphs Γ

paradigm of quant. sets

$$\text{set } X \leftrightarrow \text{algebra } C(X) = \{f: X \rightarrow \mathbb{C}\} \simeq \mathbb{C}^{|X|}$$

$\leadsto$ we should take $B = C(V_\Gamma)$, V_Γ : vertex set of Γ

"standard" choice of ψ : $\psi(f) = \frac{1}{|V_\Gamma|} \sum_{v \in V_\Gamma} f(v)$
(uniform probability measure)

what are the quant. adj. matrices for this (B, ψ) ?

$$\eta: \mathbb{C} \rightarrow L^2(B) \text{ is } \lambda \mapsto \lambda \Lambda(1_{V_\Gamma})$$

$$\eta^*: L^2(B) \rightarrow \mathbb{C} \text{ is } \Lambda(f) \mapsto \psi(f)$$

$$m: L^2(B) \otimes L^2(B) \rightarrow L^2(B) \text{ is } \Lambda(\delta_v) \otimes \Lambda(\delta_m) \mapsto \delta_{v,m} \Lambda(\delta_v)$$

Claim $m^*: L^2(B) \rightarrow L^2(B) \otimes L^2(B)$ is

$$\Lambda(\delta_v) \mapsto |V_\Gamma| \Lambda(\delta_v) \otimes \Lambda(\delta_v)$$

$$\because \text{ solve } (\Lambda(\delta_v) | m(\Lambda(\delta_{v'}) \otimes \Lambda(\delta_m))) = (m^*(\Lambda(\delta_v)) | \Lambda(\delta_{v'}) \otimes \Lambda(\delta_m))$$

$$(\text{use } (\Lambda(\delta_v) | \Lambda(\delta_m)) = \frac{1}{|V_\Gamma|} \delta_{v,m})$$

$$\text{Cor. } m^* \eta: \lambda \mapsto |V_\Gamma| \lambda \sum_v \Lambda(\delta_v) \otimes \Lambda(\delta_v)$$

Now represent $X \in \mathcal{B}(L^2(B, \psi))$ as a matrix $(X_{vm})_{v,m}$ for the basis $(\Lambda(\delta_v))_{v \in V_\Pi}$; $X_{vm} = (\Lambda(\delta_v) | X | \Lambda(\delta_m))$

Prop.  =  with $Z_{vm} = |V_\Pi| X_{vm} Y_{vm}$

Proof. direct comp. with above formulas for m^* , X_{vm} , m .

Then a quant. adj. mat. op. $A \in \mathcal{B}(L^2(B))$ is:

$$\text{Diagram: two boxes A connected by a loop} = \text{Diagram: box A with a vertical line through it} \Leftrightarrow |V_\Pi| A_{vm}^2 = A_{vm} \Leftrightarrow A_{vm} = 0, \frac{1}{|V_\Pi|} \quad ($$

$$\text{Diagram: box A with a loop and a vertical line through it} = \text{Diagram: box A with a vertical line through it} \Leftrightarrow A_{mv} = A_{vm}$$

$$\text{Diagram: box A with a loop} = | \Leftrightarrow A_{vv} = \frac{1}{|V_\Pi|}$$

i.e. we get the conditions for undirected graphs

with $\begin{cases} \bullet \text{ at most one edge between } \forall v, m \in V_\Pi \\ \bullet \text{ one loop at each } v \end{cases}$



rem $|V_\Pi| A - \text{Id}$ corresp. to simple graph. (no loops)

Ex. complete graph K_n (n vertices, $v \sim m$ for all $v \neq m$)

$\rightarrow$ our convention of corresp. (quant.) adj. mat. is

$$A \Lambda(\delta_v) = \sum_{m=1}^n \Lambda(\delta_m) = |V_\Pi| \cdot \psi(\delta_v)$$

$$\text{rem. } |V_\Pi| \text{id}_{L^2(B)} = m m^*$$