

So far : a quantum graph is given by

- fin dim C^* -alg $B \cong \bigoplus_{k=1}^n M d_k$
- faithful positive functional $\psi : B \rightarrow \mathbb{C}$
correction : we allow $\psi(1_B) \neq 1$.
- quantum adjacency matrix $A \in \mathcal{B}(L^2(B, \psi))$

$$A = A^*, \quad \begin{array}{c} \text{---} \square \text{---} \square \text{---} \\ \text{---} \square \text{---} \end{array} = \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \square \end{array} = \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \square \end{array}, \quad \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \square \end{array} = |$$

for $\eta = m$: prod. of B , $\downarrow$: unit $\mathbb{C} \rightarrow B$, ...

up to $B \cong L^2(B, \psi)$ as lin sp.

$$\text{and } \psi = m^*, \quad \eta = \eta^*$$

Rem when we scale $\psi \rightsquigarrow \psi' = r\psi$ ($r > 0$)

$$m_{\psi'}^*(\Lambda_{\psi'}(a)) = r^{-1} \sum \Lambda_{\psi'}(x_i) \otimes \Lambda_{\psi'}(y_i) \text{ for } m_{\psi}^*(\Lambda_{\psi}(a)) = \sum_i \Lambda_{\psi}(x_i) \otimes \Lambda_{\psi}(y_i)$$

$\uparrow$ computed on $L^2(B, \psi')$

$$\text{from } (m_{\psi'}^*(\Lambda_{\psi'}(a)) \mid \Lambda_{\psi'}(b) \otimes \Lambda_{\psi'}(c))$$

$$= (\Lambda_{\psi'}(a) \mid m(\Lambda_{\psi'}(b) \otimes \Lambda_{\psi'}(c))) = (\Lambda_{\psi'}(a) \mid \Lambda_{\psi'}(bc))$$

$$= \psi'(a^*bc) = r\psi(a^*bc) = r(m_{\psi}^*(\Lambda_{\psi}(a)) \mid \Lambda_{\psi}(b) \otimes \Lambda_{\psi}(c))$$

$$\text{and } (\Lambda_{\psi'}(x) \otimes \Lambda_{\psi'}(y) \mid \Lambda_{\psi'}(b) \otimes \Lambda_{\psi'}(c)) = r^2 (\Lambda_{\psi}(x) \otimes \Lambda_{\psi}(y) \mid \Lambda_{\psi}(b) \otimes \Lambda_{\psi}(c))$$

Ex. V : finite set ("vertex set") $\rightarrow B = C(V) \cong \mathbb{C}^{|V|}$

$$\psi(f) = \sum f(v) \quad \text{satisfies } \psi(1_B) = |V|.$$

q. a. m. for ψ : $A = (a_{vm})_{v,m \in V}$ s.t.

$$a_{v,m}^2 = a_{v,m} = a_{m,v}, \quad a_{vv} = 1.$$

cf. computation for the state $\frac{1}{|V|}\psi$ from last time

Def. the complete quantum graph on (B, ψ) is

given by the quant. adj. mat. $A(\Lambda_{\psi}(a)) = \psi(a) \Lambda_{\psi}(1_B)$

i.e. A is the rank 1 op. $\eta \circ \eta^*$

(cont) graphically : $A = \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$

we check the conditions for quant. adj. mat. as

$$\begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = 1$$

(1 is the unit $\Rightarrow \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = 1$)

classically : with $B = C(V)$, $\psi(f) = \sum_{v \in V} f(v)$

$$A(f) = \left(\sum_v f(v) \right) 1_B \quad A = \begin{bmatrix} 1 & \dots & 1 \\ 1 & \dots & 1 \end{bmatrix}$$

$\leadsto$ the compl. graph K_n ($n = |V|$) up to



Def. the empty quant. graph on (B, ψ) is

given by $A = (m m^*)^{-1}$.

to check

$$\begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$$

we verify

$$\begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array} (m m^*)$$

Key Fact :

$$\begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$$

this allows

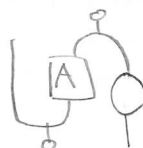
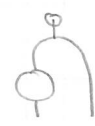
$$\begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$$

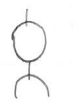
associativity

similar comp. for bottom half.

$$\begin{array}{|c|} \hline \bullet \\ \hline \end{array} = \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$$

to check  = 

verify  = $|$ using  = 

the latter:  =  =  = 
key.

Proof of Key Fact  =  = 

(this means $m^* : B \rightarrow B \otimes B$ is a homomorphism of B -bimodules.

we check $m^*(\Lambda(a)) = a m^*(\Lambda(1)) = m^*(\Lambda(1)) a$
 $(a \Lambda(b) = \Lambda(ab) = \Lambda(a) b)$

$(m^* \Lambda(a) \mid \Lambda(b) \otimes \Lambda(c)) = \psi(a^* b c)$

$(a m^*(\Lambda(1)) \mid \Lambda(b) \otimes \Lambda(c)) = (m^*(\Lambda(1)) \mid a^* \Lambda(b) \otimes \Lambda(c))$
 $= (m^*(\Lambda(1)) \mid \Lambda(a^* b) \otimes \Lambda(c)) = \psi(a^* b c)$

first eq: $(a \Lambda(x) \mid \Lambda(y)) = \psi(x^* a^* y)$
 $= (\Lambda(x) \mid a^* \Lambda(y))$

$(m^*(\Lambda(1)) a \mid \Lambda(b) \otimes \Lambda(c)) = (m^*(\Lambda(1)) \mid \Lambda(b) \otimes \Lambda(c) \sigma(a^*))$
 $= \psi(b c \sigma(a^*)) = \psi(a^* b c)$

σ : automorphism of B s.t. $\psi(xy) = \psi(y \sigma(x))$

when $B = M_d$, $\psi(x) = \text{Tr}(Qx)$
 $\sigma(x) = Q x Q^{-1}$

