

Quantum graphs from commutative groups

G : finite commutative group, $X \subset G$ subset.

$\leadsto$ Cayley graph for (G, X) (write $\text{Cay}(G, X)$)

- vertex set

- $\exists$ edge between a and $b \Leftrightarrow ab^{-1} \in X$.

$e_G \in X \Leftrightarrow \exists$ loop at each vertex.
unit elem

Goal: deform $C(G)$ to a noncommutative C^* -algebra B and get a quant. adj. mat corresponding to $\text{Cay}(G, X)$

Dual group $\hat{G} = \text{Hom}(G, \mathbb{T})$ for $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$

$\hat{G} \cong G$, but not canonically

when $G = \mathbb{Z}/n$ $\varphi \in \hat{G}$ determined by $\varphi([1]) = \zeta, \zeta^n = 1$

Prop. $C(G) \cong \mathbb{C}[\hat{G}]$ complex group ring of $\hat{G}$.

ie. $C(G)$ is linearly spanned by $\varphi \in \hat{G}$

group law of $\hat{G} \equiv$ prod as functions on G .

$\rightarrow$ 2-cocycle deformation of group rings

H : group ($= \hat{G}$ for us)

$\omega: H \times H \rightarrow \mathbb{T}$ 2-cocycle

$$\omega(h_1, h_2, h_3) \omega(h_1, h_2) = \omega(h_1, h_2 h_3) \omega(h_2, h_3)$$

$\leadsto$ twisted group ring $\mathbb{C}_\omega[H]$: same lin sp. $\mathbb{C}[H]$

new prod $h_1 \circ_\omega h_2 = \omega(h_1, h_2) h_1 h_2$.

(we may assume $\omega(e_H, h) = 1 = \omega(h, e_H) = \omega(h, h^{-1})$)

standard pos. func. $\psi: C(H) \rightarrow \mathbb{C}$ $\psi(f) = \sum_{h \in H} f(h)$

becomes $\psi: \mathbb{C}[\hat{H}] \rightarrow \mathbb{C}$, $\psi(\varphi) = \delta_{e_{\hat{H}}, \varphi} |H|$.

Example $H = (\mathbb{Z}/2)^2$ $\omega(h, h') = (-1)^{h_1 h'_2 + h_2 h'_1}$

(or $\omega(h, h') = h_1 h'_2 - h_2 h'_1$ for $\omega^n = 1$ on $(\mathbb{Z}/n)^2$)

$\Rightarrow \mathbb{C}_\omega[H] \cong M_2$ ($\omega = \text{prim rt of } 1 \rightarrow \mathbb{C}_\omega[H] \cong M_n$)

m^* for $(\mathbb{C}_\omega[H], \psi_{\text{std}})$: $m^*(h) = \frac{1}{|H|} \sum_{k \in H} \overline{\omega(k^{-1}, kh)} k^{-1} \otimes kh$

so $mm^* = \text{id}$ on $L^2(\mathbb{C}_\omega[H], \psi_{\text{std}})$

Adjacency matrix of $\text{Cay}(G, X)$ ($A_{g,h} = \delta_{gh^{-1} \in X}$)

as lin. map on $\mathbb{C}[\hat{G}]$: $\varphi \mapsto \alpha_\varphi \cdot \varphi$ $\alpha_\varphi \in \mathbb{R}$

$$\sum_{\varphi'} \alpha_{\varphi'^{-1}} \alpha_{\varphi' \varphi} = |H| \alpha_\varphi, \quad \alpha_{\varphi^{-1}} = \alpha_\varphi, \quad \sum_{\varphi} \alpha_\varphi = |H|$$



Prop. $A_\omega \in B(L^2(\mathbb{C}_\omega[\hat{G}], \psi_{\text{std}}))$, $A_\omega \wedge(\varphi) = \alpha_\varphi \wedge(\varphi)$

defines a quant. adj. mat. on $(\mathbb{C}_\omega[\hat{G}], \psi_{\text{std}})$

Proof $\begin{matrix} m \\ \swarrow \end{matrix} \begin{matrix} \boxed{A_\omega} & \boxed{A_\omega} \\ \searrow & \swarrow \end{matrix} \begin{matrix} m^* \\ \swarrow \end{matrix}$ is still $\wedge(\varphi) \mapsto \frac{1}{|G|} \sum \alpha_{\varphi'^{-1}} \alpha_{\varphi' \varphi} \varphi$

Example $G = (\mathbb{Z}/2)^2$, $X = \{[0,0], [0,1], [1,0]\}$

$$\alpha_{[0,0]} = 3, \quad \alpha_{[1,0]} = \alpha_{[0,1]} = 1$$

$$\alpha_{[1,1]} = -1$$

