

So far: we defined quantum graphs for (B, ψ)
via quant. adj. mat. $\xrightarrow{\text{fin dim } C^*\text{-alg}} \uparrow \text{faithful pos.}$

in fact $\exists$ injective $\{q. \text{ adj. mat. for } (B, \psi)\}$

$\forall$ tracial
faithful pos. ψ'

$\rightarrow \{q. \text{ adj. mat. for } (B, \psi')\}$

We will prove this later (Thm 5.29 and following remark)

Sketch for commutative case: $B = C(V)$
 $\xrightarrow{\text{fin. set}}$

suppose $\psi(f) = \sum_{v \in V} p_v f(v)$ for $p_v > 0$

then $A = (A_{v,m})_{v,m \in V}$ is a quant. adj. mat. for (B, ψ)

$\Leftrightarrow A_{v,m} = 0$ or p_m , $A_{w,v} = p_v p_m^{-1} A_{v,m}$, $A_{v,v} = p_v$

$\Leftrightarrow (A_{v,m} p_m^{-1})_{v,m}$ is the adj. mat of a graph.

$\rightarrow$ Is there an invariant way to define "q.-graphs on B "?

Operator bimodule approach (Daws, Section 4)

B : fin. dim C^* -alg.

H : fin. dim Hilbert space

suppose B is faithfully represented on H

i.e. $B \xrightarrow{\pi} B(H)$ injective hom, compat. with $*$.

write $\pi(B)' = \{x \in B(H) : \forall a \in B \quad x \pi(a) = \pi(a) x\}$

(we often write B')

Def. (4.1) a quantum relation on (B, π) is

a subspace $S \subset B(H)$ s.t. $\pi(B)' S \pi(B)' \subset S$

($x \in S, y, z \in \pi(B)' \Rightarrow y x z \in S$) $\xrightarrow{\text{eq. by } Id_H \in \pi(B)}$

Def (4.2 & 4.3) a quantum graph on (B, π) is

a quant. rel. $S \subset B(H)$ which is

1.) reflexive: $\pi(B)' \subset S$ and 2.) symmetric $S^* = S$
 $\Downarrow Id_H \in S$

Rem. these defs. make sense for ∞ -dim C^* -algs
 (take weak*-closed $\pi(B)'$ -bimodules $\subset B(H)$)

Independence on π

concrete presentation of π and $\pi(B)'$

since $B \cong \bigoplus_{k=1}^n M_{d_k}$ any faithful rep. is of the

form $B \xrightarrow{\pi} B(\mathbb{C}^N) \simeq M_N$, $N = d_1 m_1 + \dots + d_n m_n$

($m_k > 0$: multiplicity)

sending $(x_k)_{k=1}^n$ to $x_1 \otimes I_{m_1} + \dots + x_n \otimes I_{m_n}$

via $\mathbb{C}^N \simeq \mathbb{C}^{d_1} \otimes \mathbb{C}^{m_1} \oplus \dots \oplus \mathbb{C}^{d_n} \otimes \mathbb{C}^{m_n}$ (

then $\pi(B)' \simeq \bigoplus_{k=1}^n M_{m_k}$

given another $\pi' : B \rightarrow B(H')$, mult. $(m'_k)_{k=1}^n$

we have $H' \cong K \otimes_{\pi'(B)} H$, $K = \bigoplus_{k=1}^n \mathbb{C}^{m'_k} \otimes \mathbb{C}^{m_k}$

$K : \pi'(B)' - \pi(B)'$ -bimodule

given a quant. rel. $S \subset B(H)$ on (B, π)

we get $S' = \text{span } B(\mathbb{C}^{m_k}, \mathbb{C}^{m'_k}) \subset B(\mathbb{C}^{m'_e}, \mathbb{C}^{m_e})$
 $\subset B(H')$ quant. rel. on (B, π')

so $S' \simeq K \otimes_{\pi(B)'} S \otimes_{\pi(B)'} K^*$

this corresp. $S \mapsto S'$ defines a bijection

$\{ \text{q. rel. on } (B, \pi) \} \rightarrow \{ \text{q. rel. on } (B, \pi') \}$

compat. with q. graphs.

Examples 1. empty quant. graph. $S = \pi(B)'$

2. complete quant. graph $S = B(H)$

Classical case $B = C(V)$ (V : fin. set)

take $H = \ell^2(V)$, $\pi : B \rightarrow B(\ell^2(V))$ action by mult.
(GNS representation for ψ_{std}).

Prop. $\pi(B)' = \pi(B)$ i.e. $\pi(B) \subset B(\ell^2(V))$ is
maximally abelian subalg.

Proof. the std basis $(\delta_v)_{v \in V}$ consists of the
joint eigenvectors for $\pi(f)$ $f \in B$

$x \in \pi(B)' \Rightarrow$ each δ_v is an eigenvector of x .
 $\Rightarrow x = \pi(f)$ for $x \delta_v = f(v) \delta_v$ $\square$

Prop. any quant. rel. $S \subset B(\ell^2(V))$ on (B, π) is
of the form $S_E = \{ (x_{vm})_{v,m \in V} : x_{vm} = 0 \text{ if } (v,m) \notin E \}$
for some subset $E \subset V \times V$.

Proof set $E_S = \{ (v,m) \in V \times V : \exists x \in S \quad \delta_v x \delta_m \neq 0 \}$
then $S = S_{E_S}$ from $S = \pi(B) S \pi(B)$ $\square$

Cor. any quant. graph $S \subset B(\ell^2(V))$ on (B, π)
is of the form S_E for the edge set $E \subset V \times V$
of an undirected graph with loops around vertexes

Proof $S^* = S$ means $(v,m) \in E_S \Leftrightarrow (m,v) \in E_S$

$\text{Id}_{\ell^2(V)} \in S$ means $(v,v) \in E_S$ for all $v \in V$ $\square$