

Understanding "m (A \otimes A) m^* = A" for quant. adj.
mat. ops.

B : fin. dim C^*-alg, $\psi: B \rightarrow \mathbb{C}$ faith. pos. func.

$a, b \in B \rightsquigarrow \theta_{\Lambda(a), \Lambda(b)} \in \mathcal{B}(L^2(B, \psi))$ defined by

$$\theta_{\Lambda(a), \Lambda(b)}(\Lambda(c)) = (\Lambda(a) | \Lambda(c)) \Lambda(b)$$

Notation $B^{\text{op}} = \{a^{\text{op}} : a \in B\}$ opposite algebra.

$$a^{\text{op}} b^{\text{op}} = (ba)^{\text{op}}.$$

(B^{op} is isom. to B , but we do this for "correct" bookkeeping)

$\Sigma: B \otimes B^{\text{op}} \rightarrow B \otimes B^{\text{op}}, a \otimes b^{\text{op}} \mapsto b \otimes a^{\text{op}}$ "flip"

$\mathbb{F}: \mathcal{B}(L^2(B, \psi)) \rightarrow B \otimes B^{\text{op}}$ linear map extending
 $\theta_{\Lambda(a), \Lambda(b)} \mapsto a^* \otimes b^{\text{op}}$

Rem $\mathbb{F}$ is a linear isom.

Prop (5.3) Given $A \in \mathcal{B}(L^2(B, \psi))$

$$1. \mathbb{F}(A^*) = \Sigma(\mathbb{F}(A)^*)$$

$$2. \mathbb{F}\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \boxed{A} \begin{array}{c} \text{---} \\ \text{---} \end{array}\right) = (\text{id} \otimes \sigma_i) \Sigma(\mathbb{F}(A)) \quad \text{see below for } \sigma_i$$

$$3. \mathbb{F}\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \boxed{A} \begin{array}{c} \text{---} \\ \text{---} \end{array}\right) = (\sigma_i \otimes \text{id}) \Sigma(\mathbb{F}(A))$$

$$4. \mathbb{F}\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \boxed{A_1} \boxed{A_2} \begin{array}{c} \text{---} \\ \text{---} \end{array}\right) = \mathbb{F}(A_2) \mathbb{F}(A_1)$$

Baby modular theory:

Fix a tracial faithful pos. func $\text{Tr}: B \rightarrow \mathbb{C}$

(when $B \cong \bigoplus_{k=1}^n M_{d_k}$ take $\text{Tr}((a_k)_k) = \sum p_k \text{Tr}_{M_{d_k}}(a_k)$, $p_k > 0$)

Given ψ , $\exists!$ $Q \in B$ s.t. $\psi(a) = \text{Tr}(Qa)$.

Q is positive: $Q = b b^*$ for some $b \in B$

Put $\sigma_z(a) = Q^{iz} a Q^{-iz}$ ($z \in \mathbb{C}$) autom on $\mathcal{B}$

We look at $\sigma_{\pm i}(a) = Q^{\mp 1} a Q^{\pm 1}$ in particular

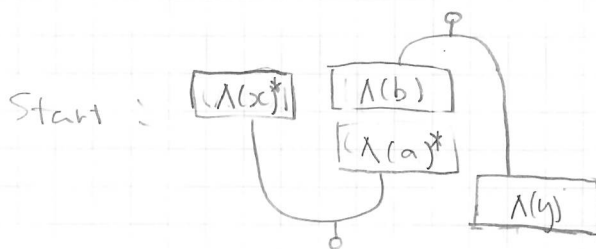
Rem. $\psi(ab) = \psi(b \sigma_i(a))$ from $\text{Tr}(Qab) = \text{Tr}(QbQaQ^{-1})$

Proof of Prop. take $A = \Theta_{\Lambda(a), \Lambda(b)}$ so $\Xi(A) = a^* \otimes b^{op}$

1. $A^* = \Theta_{\Lambda(b), \Lambda(a)} \xrightarrow{\Xi} b^* \otimes a^{op} = \Sigma(\Xi(A)^*)$

2. Put $\overline{\Xi} = \left(\begin{array}{c} \text{ } \\ \boxed{A} \end{array} \right)$;

We need to check $(\Lambda(x) | \overline{\Xi} \Lambda(y)) = (\Lambda(x) | \Theta_{\Lambda(b^*), \Lambda(\sigma_{-i}(a^*))} \Lambda(y))$
 $= (\Lambda(x) | \Lambda(\sigma_{-i}(a^*))) (\Lambda(b^*) | \Lambda(y))$



$= \overline{\psi(xa)} \psi(by)$

$\overline{\psi(xa)} = \overline{\text{Tr}(Qxa)} = \text{Tr}(a^* x^* Q) = \text{Tr}(Q x^* Q a^* Q^{-1})$
 $= (\Lambda(x) | \Lambda \sigma_{-i}(a^*))$

3. similar

4. $= (\Lambda(a_1, a_2) | \Lambda(c)) \Lambda(b_1, b_2)$

we get $\Theta_{\Lambda(a_1, a_2), \Lambda(b_1, b_2)}$

apply Ξ and get $(a_1, a_2)^* \otimes (b_1, b_2)^{op} = a_2^* a_1^* \otimes b_2^{op} b_1^{op}$