

$$\Theta_{\Lambda(a), \Lambda(b)} \mapsto a^* \otimes b \circ p$$

satisfies $\bullet \quad \mathbb{E}(A^*) = \sum_{\text{flip}} \mathbb{E}(A)^*$

$$\bullet \quad \underline{\text{F}}\left(\begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array}\right) = (\text{id} \otimes \underbrace{\sigma_{-1}}_{Q \sim Q^{-1}}) \Sigma \underline{\text{F}}(A)$$

$$\mathbb{E} \left(\begin{array}{|c|c|} \hline A_1 & A_2 \\ \hline \end{array} \right) = \mathbb{E}(A_2) \mathbb{E}(A_1)$$

Recall: $\sigma_2(a) = Q^{iZ} a Q^{-iZ}$ for $\psi(a) = \text{Tr}(Q a)$

We take the modified map $\mathbb{I}_{0,1/2} = (\text{id} \otimes \sigma_{i/2}) \mathbb{I}$

Thm (5.17) $\mathbb{F}_{0,1/2}$ gives a bijective corresp.

$$\{ A \in \mathcal{B}(L^2(B, \psi)) : A = A^*, \text{ [diagram of } A \text{ as a box with two external legs]} \} = \{ \text{[diagram of } A \text{ as a box with two external legs]} \}$$

$$\longleftrightarrow \{ e \in B \otimes B^{op} : e = e^* = e^2 = \sum (e) = (\sigma_{\bar{z}} \otimes \sigma_{\bar{z}})(e) \quad (\bar{z} \in \mathbb{C}) \}$$

$$(\sigma_z \text{ on BOP} : (Q^{iz} \text{ and } Q^{-iz})^{OP})$$

Proof. Generally

$$1. \quad \mathbb{F}_{0,1/2}(A^*) = (\sigma_{1/2} \otimes \sigma_{1/2}) \sum \mathbb{F}_{0,1/2}(A)^*$$

$$(u_{Se} \quad \sigma_{i/2}(a))^* = (Q^{-1/2} \ a \ Q^{1/2})^* = Q^{1/2} \ a^* \ Q^{-1/2} = \sigma_{-i/2}(a^*)$$

$$2. \quad \Phi_{0,1/2} \left(\begin{array}{c} \text{ } \\ \text{ } \circ \\ \boxed{A} \\ \text{ } \circ \end{array} \right) = (\sigma_{-1/2} \otimes \sigma_{-1/2}) \Sigma \Phi_{0,1/2}(A)$$

$$3. \quad \mathbb{F}_{0,1/2} \left(\begin{array}{|c|} \hline A_1 \\ \hline \end{array} \begin{array}{|c|} \hline A_2 \\ \hline \end{array} \right) = \mathbb{F}_{0,1/2}(A_2) \mathbb{F}_{0,1/2}(A_1)$$

From A as in the claim, we get $e = \mathbb{F}_{0,1,2}(A)$

s.t. $e = e^2 = (\sigma_{i/2} \otimes \sigma_{i/2})(\Sigma(e^*)) \underset{(\#)}{=} (\sigma_{i/2} \otimes \sigma_{i/2})(\Sigma(e))$

(#) implies this is inv. under $*$. (use $\sigma_{i/2}(a^*) = \sigma_{i/2}(a)^*$)

Then (ii) becomes
$$e = (\sigma_i \otimes \sigma_i)(e) = (Q^{-1} \otimes Q^{\text{op}})e(Q \otimes (Q^{-1})^{\text{op}})$$

then e comm. with any $f(Q^{\frac{-1}{2}} \otimes Q^{\text{op}})$ $f(x)$ holom.

For $f(x) = x^{-iz}$ we set $(\sigma_z \otimes \sigma_z)(e) = e$

At $z = \pm 1/2$ we get $e = \Sigma(e)$ $\square$

Next: given faithful rep $\pi: B \rightarrow B(H)$

relate $e \in B \otimes B^{op}$ as above to $S \in B(H)$

Intermediate obj: subsp. $V \subset H \otimes \bar{H}$

Recall $\bar{H} = \{ \langle v | : v \in H \} \simeq B(H, \mathbb{C})$

• $B(H) \simeq H \otimes \bar{H}$ by $\theta_{v,m} \leftrightarrow m \otimes \langle v |$

• $\pi^{op}: B^{op} \rightarrow B(\bar{H})$, $\pi^{op}(a^{op}) \langle v | = \langle v | \pi(a)$
 $(\pi^{op}(B^{op}))' = (\pi(B)')^{op}$

We get a rep. $\pi \otimes \pi^{op}: B \otimes B^{op} \rightarrow B(H \otimes \bar{H})$

so e def's $V = (\pi \otimes \pi^{op})(e) (H \otimes \bar{H})$

Claim $V = (\pi(B)' \otimes \pi^{op}(B^{op})') V$

use $\pi(B)' \otimes \pi^{op}(B^{op})' = (\pi \otimes \pi^{op})(B \otimes B^{op})'$

Claim $V = J_0(V)$ for $J_0(v \otimes \langle m |) = m \otimes \langle v |$

use $e = \sum e \Rightarrow (\pi \otimes \pi^{op})(e) = J_0(\pi \otimes \pi^{op})(e) J_0$

Claim $V = (\pi(Q) \otimes \pi^{op}((Q^{-1})^{op})) V$

use $(\pi \otimes \pi^{op})(\sigma_i(e)) = (\pi(Q) \otimes \pi^{op}((Q^{-1})^{op})) (\pi \otimes \pi^{op})(e)$
 $\times (\pi(Q^{-1}) \otimes \pi^{op}(Q^{op}))$

Corresponding conditions for $S \in B(H)$.

$$S = \pi(B)' S \pi(B)' = S^* = \pi(Q) S \pi(Q^{-1})$$

What about $\boxed{A} = |id$?

Lemma 5.24 | When $A = \theta_{\Lambda(a), \Lambda(b)}$ we have

$$\boxed{A} = \boxed{\Lambda(ba^*c)}$$

Take an orthonormal basis $(\Lambda(x_i))_i$ in $L^2(B, \phi)$

so $id = \sum_i \theta_{\Lambda(x_i), \Lambda(x_i)}$

cont.)

$$\begin{aligned}
 \boxed{A} &= \sum_i \frac{\Lambda(b)}{\Lambda(a)^*} \frac{\Lambda(x_i)}{\Lambda(x_i)^*} = \sum_i \frac{(\Lambda(ax_i) / \Lambda(c))}{\psi(x_i^* a^* c^*)} \frac{\Lambda(bx_i)}{b \Lambda(x_i)} \\
 &= \sum_i (\Lambda(x_i) / a^* \Lambda(c)) b \Lambda(x_i) = b a^* \Lambda(c)
 \end{aligned}$$

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Prop. $A \in \mathcal{B}(L^2(B, \psi))$, $\pi: B \rightarrow \mathcal{B}(H)$, $S \subset \mathcal{B}(H)$
as in Thm 5.17.

Then $\boxed{A} = 1 \Leftrightarrow \pi(Q^{-1/2}) \in S$

Proof. Step 1. Writing $A = \sum_i \theta_{\Lambda(a_i), \Lambda(b_i)}$

$$\boxed{A} \Leftrightarrow \sum_i b_i a_i^* = 1 \quad (\text{by Lem})$$

Step 2. $(v_j)_j$ orthonormal basis of H .

$$\hat{1} = \sum_j v_j \otimes (v_j | \in H \otimes \bar{H} \quad (\text{corr. to } \text{id}_H \in \mathcal{B}(H))$$

$$\forall v, w \in H: (v | \pi(b a^*) w) = (\hat{1} | (\pi(a^*) \otimes \pi^{op}(b^{op}))(w \otimes (v |))$$

$$\therefore \text{compute using } ((v_j | | \pi^{op}(b^{op})(v |) = (v | \pi(b) v_j)$$

$$\text{Step 3} \quad \sum_i b_i a_i^* = 1 \Leftrightarrow \sum_i (\pi \otimes \pi^{op})(a_i \otimes b_i^{* op}) \hat{1} = \hat{1}$$

from Step 2

$$\text{Step 4. } (\pi \otimes \pi^{op})(e) \text{ preserves } (\pi \otimes \pi^{op})(1 \otimes Q^{-1/2}) \hat{1}$$

$$\Leftrightarrow \sum (\pi \otimes \pi^{op})(a_i \otimes b_i^{* op}) \text{ pres. } \hat{1}$$

$$\therefore e = \sum_i a_i^* \otimes \sigma_{i/2}(b_i)^{op} = \sum_{e=e^*} a_i \otimes \sigma_{i/2}(b_i^*)^{op}$$

$$\text{goes to } \sum \pi(a_i) \otimes \pi^{op}(Q^{-1/2}) \pi^{op}(b_i^{* op}) \pi^{op}(Q^{1/2})$$