

So far: quant. adj. mat  $A \in B(L^2(B, \varphi))$

faithful rep.  $\pi: B \rightarrow B(H)$

$\leadsto$  subsp.  $S \subset B(H)$  s.t.

$$- \pi(B)' S \pi(B)' = S$$

$$- S = S^*$$

$$- \pi(Q) S \pi(Q)^{-1} = S, \quad \pi(Q^{-1/2}) \in S$$

$$(\varphi(a) = \text{Tr}(Qa)).$$

We can reverse this corresp.:  $S \subset B(H) \leftrightarrow V \subset H \otimes \bar{H}$

$$- \pi(B)' S \pi(B)' = S \text{ becomes } (\pi(B)' \otimes \pi^{op}(B^{op})') V = V$$

$$\leadsto V = E(H \otimes \bar{H}) \text{ for some } E \in \underbrace{(\pi(B)' \otimes \pi^{op}(B^{op})')}'_{\text{projection}} \text{ eq. to } \pi(B) \otimes \pi^{op}(B^{op})$$

$$- S = S^* \text{ becomes } V = J_0 V \text{ see last time}$$

$$- \pi(Q) S \pi(Q)^{-1} = S \Leftrightarrow E \text{ comm. with } \pi(Q) \otimes \pi^{op}(Q^{-1/2})$$

$$\Leftrightarrow E \text{ is invariant under } \sigma_z \otimes \sigma_z$$

$$- \pi(Q^{-1/2}) \in S \text{ becomes } m \sum e = 1 \Leftrightarrow \text{Tr}(A) = 1.$$

When  $\varphi$  is a trace (can be diff. from  $\text{Tr}$ )

$$Q = \bigoplus p_k \text{Tr}|_{M_{d_k}} \text{ on } B = \bigoplus M_{d_k}, \quad p_k > 0$$

$$\sigma_z = \text{id}, \quad \pi(Q) = \pi(B) \cap \pi(B)'$$

$$\leadsto \text{we have } \pi(B) \cap \pi(B)' = S, \quad S = S^* \neq 1.$$

operator system.

In general, if  $S_A \subset B(H)$  corresp. to  $A$  for  $(B, \varphi)$

$$S_0 = \pi(Q)^{1/4} S_A \pi(Q)^{1/4} \text{ is an op system}$$

$\pi(B)'$ -bimodule

$\leadsto$  for any trace  $\varphi_0: B \rightarrow \mathbb{C}$  we get a

quant. adj. mat  $A_0$  for  $(B, \varphi_0)$

We get an injective map

$$\begin{aligned} & \{ \text{quant adj. mat for } (B, \psi) \} \\ & \longrightarrow \{ \text{quant adj. mat for } (B, \psi_0) \} \\ & \quad \text{tr.} \end{aligned}$$

(correcting the claim on Jan 30)

More on conventions

when  $A = A^*$

•  $\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} = \boxed{A}$  iff  $\begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} = \boxed{A}$  vertical flip is  $T \mapsto T^*$

•  $\begin{array}{c} | \\ \boxed{A} \\ | \end{array} = |$  iff  $\begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} = |$  consider  $\begin{array}{c} | \\ \boxed{T} \\ | \end{array} \mapsto \begin{array}{c} \circ \\ | \\ \boxed{T} \\ | \\ \circ \end{array}$  and

with above zigzag  $\begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} = \boxed{A} \Rightarrow \begin{array}{c} | \\ \boxed{A} \\ | \end{array} = \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array}$  (see next page)

Another def. (Section 3)

Completely positive map  $A : B \rightarrow B$

such that  $m(A \otimes A)m^* = A$  (up to  $B \cong L^2(B, \psi)$ )

CP maps  $A : B \rightarrow B \iff$  positive elems  $a \in B \otimes B^{op}$

$$(\theta_{\psi_0, c} : b \mapsto \psi(ab)c) \iff c \otimes \sigma_{-1/2}(a)^{op} = c \otimes (Q^{1/2} a Q^{-1/2})^{op}$$

$$m(A \otimes A)m = A \iff a \text{ is idempotent. } a = a^2$$

undirected  $\begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} = \boxed{A}$ , has loops at verts

$$\begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} = | \dots$$

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$$\left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right) = \boxed{A} \quad \text{implies} \quad \left( \begin{array}{c} | \\ \boxed{A} \\ | \\ \circ \end{array} \right) = | \Leftrightarrow \left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right) = |$$

Step 1.  $\left( \begin{array}{c} | \\ \boxed{A} \\ | \\ \circ \end{array} \right) = \left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right)$ ; apply  $\boxed{T} \mapsto \left( \begin{array}{c} | \\ \boxed{T} \\ | \\ \circ \end{array} \right)$  to  
the assumption, use  $\left( \begin{array}{c} \circ \\ | \\ \circ \end{array} \right) = |$  (by Frob. alg prop.)

Step 2

$$\left( \begin{array}{c} \circ \\ | \\ \left( \begin{array}{c} | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \\ | \\ \circ \end{array} \right) = \left( \begin{array}{c} | \\ \left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \\ | \\ \circ \end{array} \right)$$

use (co)associativity for  $m^*$  to get

$$\begin{aligned} \text{LHS} &= \left( \begin{array}{c} \circ \\ | \\ \left( \begin{array}{c} | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \\ | \\ \circ \end{array} \right) \stackrel{\text{A.1}}{=} \left( \begin{array}{c} \circ \\ | \\ \left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \\ | \\ \circ \end{array} \right) \stackrel{\text{Fr.}}{=} \left( \begin{array}{c} \circ \\ | \\ \left( \begin{array}{c} | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \\ | \\ \circ \end{array} \right) \stackrel{\text{Fr.}}{=} \left( \begin{array}{c} \circ \\ | \\ \left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \\ | \\ \circ \end{array} \right) \\ &= \left( \begin{array}{c} | \\ \left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \\ | \\ \circ \end{array} \right) = \left( \begin{array}{c} | \\ \left( \begin{array}{c} | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \\ | \\ \circ \end{array} \right) \stackrel{\text{Fr.}}{=} \left( \begin{array}{c} | \\ \boxed{A} \\ | \\ \circ \end{array} \right) = \left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right) \end{aligned}$$

Step 3.  $\left( \begin{array}{c} | \\ \boxed{A} \\ | \\ \circ \end{array} \right) = | \Leftrightarrow \left( \begin{array}{c} \circ \\ | \\ \boxed{A} \\ | \\ \circ \end{array} \right) = |$

apply  $\boxed{T} \mapsto \left( \begin{array}{c} | \\ \boxed{T} \\ | \\ \circ \end{array} \right)$  and use Step 2.