

Recall from the first lecture

quantum channel $N : B(H_I) \rightarrow B(H_O)$

$$\rho \mapsto \sum K_i \rho K_i^*$$

for $K_i \in B(H_I, H_O)$, $\sum K_i^* K_i = \text{id}_{H_I}$.

$\leadsto S = \text{span} \langle K_i^* K_j \rangle \subset B(H_I)$ as analogue of confusability graph

We want to generalize this for more general setting. (completely positive maps between C^* -algebras).

It's more convenient to take unital compl. pos. maps.

($N^t : B(H_O) \rightarrow B(H_I)$, $x \mapsto \sum_i K_i^* x K_i$)

Structure of ucp maps (Stinespring dilation)

$$\begin{cases} C : (\text{fin-dim}) C^*\text{-alg.}, H : (\text{fin-dim}) \text{Hilb. sp.} \\ \theta : C \rightarrow B(H) \text{ unital compl. pos. map} \end{cases}$$

$\leadsto \exists$ Hilb. sp K , rep $\pi : C \rightarrow B(K)$,
isometry $V : H \rightarrow K$ ($V^* V = \text{id}_H$)

Such that $\theta(x) = V^* \pi(x) V$ ($x \in C$)

• Extra structure : $\rho : \theta(C)' \rightarrow B(K)$ by

$$\rho(y) \pi(x) V m = \pi(x) V y m$$

• Minimal model : $K = \text{span} \langle \pi(x) V m : x \in C, m \in H \rangle$

given another model $\tilde{K}, \hat{\pi}, \tilde{V}$

$u : K \rightarrow \tilde{K}$, $\pi(x) V m \mapsto \hat{\pi}(x) \tilde{V} m$ ($x \in C, m \in H$)

is well-defined isometry, $u V = \tilde{V}$

why: $(\pi(x) V m \mid \pi(y) V u) = (m \mid V^* \pi(x^* y) V u)$
 $= (m \mid \theta(x^* y) u)$ only dep. on x, y, u, m

• when $M_a \rightarrow M_b$
 $\theta(x) = \sum_{i=1}^n K_i^* x K_i$ for $K_i \in M_{a \times b}$

$V = \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix}$ is the corresp. isomet. $V^* V = I_b$

$\pi(x) = \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix}$, $K = \mathbb{C}^{n \times a}$

Th'm 6.4 B, C (fin-dim) C^* -algs

$\pi_0: B \rightarrow B(H)$ faithful rep

$\theta: C \rightarrow B$ ucp map.

K, π, V Stinespring dil. for $\pi_0 \circ \theta: C \rightarrow B(H)$

Then $S = V^* \pi(C)' V \subset B(H)$ is a quant graph for (B, π_0) , indep. of K, π, V .

Proof Step 1 S is an op. sys ($S = S^* \ni id_H$) follows from $\pi(C)' = \pi(C)'^* \ni id_K$, $V^* V = id_H$

Step 2 $\pi_0(B)' S \pi_0(B)' = S$

$\pi_0(B)' \subset (\pi_0 \circ \theta)(C)' \xrightarrow{p} B(K)$ by

$p(y) \pi(x) V m = \pi(x) V y m$

$p((\pi_0 \circ \theta)(C)') \subset \pi(C)'$

in particular $y V z = y p(z) V$ for $y \in \pi(C)'$
 $z \in \pi_0(B)'$

$\leadsto V^* y V z = V^* y p(z) V \in S$

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Step 3 independence on K, π, V .

take the minimal model $\hat{K}, \hat{\pi}, \hat{V}$

and comparison $u: \hat{K} \rightarrow K$ ($u\hat{V} = V$)

we also have $u\hat{\pi}(x) = \pi(x)u$ ($x \in C$)

$$\Rightarrow u^* \pi(C)' u = \hat{\pi}(C)'$$

$$\Rightarrow \hat{V}^* \hat{\pi}(C)' \hat{V} = V^* \pi(C)' V \quad \square$$

Examples:

1. $C = B$, $\theta = \text{id}$. $\Rightarrow S = \pi_0(B)'$ empty graph

2. $C = \mathbb{C}$, $\theta: \mathbb{C} \rightarrow B$ embedding as unit

$\Rightarrow S = B(H)$ complete graph.

