

Quantum channel from quantum graph

We've seen : $\theta : C \rightarrow B$ ucp, $\pi_0 : B \rightarrow B(H)$
induce a quant. graph for (B, π_0) .

($S = V^* \pi(C)' V \in B(H)$ for Stinespring
dilation $(K, \pi : C \rightarrow B(K), V : H \rightarrow K)$ of $\pi_0 \circ \theta$)

Can we go backwards? i.e. given a quant
graph, can we find θ ?

Duan '09 (arXiv:0906.2527) Yes for $B = B(H)$.
($\pi_0 = \text{id}$).

To be precise: given an op. sys $S \subset B(H)$

$\exists$ fin dim C^* -alg C , ucp map $C \xrightarrow{\theta} B(H)$

s.t. $S = V^* \pi(C)' V$ with (K, π, V) for θ .

Rem. Since $\pi_0(B)' = \mathbb{C}$, no condition about bimodule

Step 1. S has a basis $(x_i)_{i=1}^n$ with
positive x_i , $\sum x_i = \text{id}_H$.

$\because S = S^* \Rightarrow$ linearly spanned by selfadj. elems.

$$x = \frac{1}{2}(x+x^*) + i \frac{1}{2i}(x-x^*)$$

$\text{id}_H \in S \Rightarrow$ linearly spanned by pos. elems

$x = x^* \Rightarrow x + a \text{id}_H$ is pos. for
 $a \gg 0$

$\Rightarrow$ basis $(x_i)_{i=1}^n$, we may assume $\text{id}_H \geq \sum x_i$
and $\text{id}_H \notin \text{span}(x_1, \dots, x_{n-1})$

$\Rightarrow$ replace x_n by $\text{id}_H - (x_1 + \dots + x_{n-1})$
to get the claim

Step 2 Construction of K, V .

Choose $a_i \in B(H)$ s.t. $x_i = a_i^* a_i$

$$K = H^{\oplus n}, \quad V: H \rightarrow K, \quad v \mapsto a_1 v \oplus \dots \oplus a_n v.$$

$$\text{Then } V^* V v = \sum a_i^* a_i v = v.$$

$$V: \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad V^*: [a_1^* \dots a_n^*]$$

Step 3 Choice of $C, \pi: C \rightarrow B(K)$

$$C = B(H) \oplus \dots \oplus B(H), \quad \pi: \text{nat. rep.}$$

$$\text{Writing } B(K) \simeq B(H) \otimes B(\mathbb{C}^n)$$

we are taking $B(H) \otimes D_n$, $D_n = \{ \text{diag mat.} \}$

Step 4 Compute $V^* \pi(C)' V$

$$\pi(C)' = \mathbb{C} \text{id}_H \oplus \dots \oplus \mathbb{C} \text{id}_H \simeq \mathbb{C} \text{id}_H \otimes D_n$$

$$\leadsto V^* \pi(C)' V = \text{span} \langle a_i^* a_i \rangle = \text{span} \langle x_i \rangle = S \quad \square$$

Summary : find $(a_i)_{i=1}^n \subset B(H)$ s.t. $S = \text{span} \langle a_i^* a_i \rangle$

take $\Theta: B(H)^{\oplus n} \rightarrow B(H)$, $(y_i)_{i=1}^n \mapsto \sum a_i^* y_i a_i$