

Zero-error capacity of classical channels

Γ : (undirected) graph.

Def $\alpha(\Gamma)$ = independence number = $\max \{k : \exists \text{ empty subgraph of size } k \text{ in } \Gamma\}$

When F is a channel with input set I and output set O

and Γ is the confusability graph of F

$\Rightarrow \alpha(\Gamma)$ is the maximum number of different inputs we can send through F w/o error, "letters"

How about max. num. of diff. n -letter words we can send w/o errors?

$\rightarrow$ consider the channel $F^n: I^n \rightarrow O^n$

Its confusability graph is Γ^n

- vertex set $V_{\Gamma^n} = V_{\Gamma}^n$

- $(v_1, \dots, v_n) \sim (w_1, \dots, w_n) \Leftrightarrow \forall i \ v_i \sim w_i$

Rem. Our convention here is "each vertex has a loop"

Def zero-error capacity (Shannon capacity)

$$C_0(\Gamma) = \lim_n \frac{1}{n} \log \alpha(\Gamma^n) = \sup_n \frac{1}{n} \log \alpha(\Gamma^n)$$

Lovász's estimate

Def $\vartheta(\Gamma) = \max \{ \|A\| : A \in M_{V_{\Gamma}}, \text{ pos. semidef., } A_{vv} = 1, v \times w \Rightarrow A_{vw} = 0 \}$

Obs. $\alpha(\Gamma) \leq g(\Gamma)$

$$\left\{ \begin{array}{l} \bullet A = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \\ & & & 1 \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \in M_{V_\Gamma} \text{ satisfies the cond. in the def. of } g(\Gamma) \\ \bullet \underbrace{\left\| \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{bmatrix} \right\|}_{k} = k \end{array} \right.$$

vert. for empty subgraph

Th'm (Lovász, '79) $g(\Gamma_1 \times \Gamma_2) = g(\Gamma_1) g(\Gamma_2)$

Key idea: by semidefinite programming methods

$$g(\Gamma) = \min \{ \lambda_{\max}(A) : A \in M_{V_\Gamma}, A = A^*, A_{vw} = 1 \text{ for } v=w, v \neq w \}$$

RHS gives $g(\Gamma_1 \times \Gamma_2) \leq g(\Gamma_1) g(\Gamma_2)$

First def. gives $g(\Gamma_1 \times \Gamma_2) \geq g(\Gamma_1) g(\Gamma_2)$

Cor. $C_0(\Gamma) \leq \log g(\Gamma)$

Analogue for quantum channels

(Duan - Severini - Winter '13)

$N: \mathcal{B}(H_I) \rightarrow \mathcal{B}(H_O)$ trace pres. compl. pos. map.

Modelling "distinguishable states"

- as vectors: orthogonal vectors in H .

- as density matrices: density matrices

$$p_1, p_2 \in \mathcal{B}(H) \quad (p_i \geq 0, \text{Tr } p_i = 1) \quad \text{s.t. } p_1 p_2 = 0$$

$\leadsto$ "N does not confuse inputs $v, w \in H_I$ "

$$\text{means } \underbrace{N(vv^*) N(w w^*)}_{\substack{\uparrow \\ \text{rank 1 proj. to } v, w}} = 0$$

Suppose we have $N(p) = \sum K_i p K_i^*$

with $K_i \in \mathcal{B}(H_I, H_O)$, $\sum K_i^* K_i = \text{id}_{H_I}$

$$\text{Tr } N(vv^*) N(w w^*) = \text{Tr } \sum_{i,j} K_i v v^* K_i^* K_j w w^* K_j^*$$

$$\text{[Rem]} \quad N(p_1) N(p_2) = 0 \Leftrightarrow \text{Tr } N(p_1) N(p_2) = 0$$

$$\text{LHS} = \text{Tr } \sum_{i,j} \underbrace{(v^* K_i^* K_j w)}_{\substack{\text{already num.} \\ \text{we can remove}}} (\text{its adjoint})$$

$$= \sum_{i,j} |v^* K_i^* K_j w|^2$$

$\Rightarrow$ we set $v^* K_i^* K_j w = 0$ for all i, j .

i.e. $\underbrace{v^* S w}_{} = 0$ for $S = \text{span} \langle K_i^* K_j \rangle$.

$S \perp w v^*$ for the trace inn. prod.

Def $\alpha(S) = \max \{k : \exists \text{ orth. fam. of vecs } v_1, \dots, v_k \text{ indep. num. s.t. } S \perp v_i v_i^* \text{ } i \neq j \}$

variations

Def $\alpha_q(S) = \max \{ \dim H_{I'} : H_{I'} \subset H_I$

quant. indep.
num

$P_{I'} S P_{I'} = \mathbb{C} P_{I'}$ "error correction"
 $\exists D : \mathcal{B}(H_O) \rightarrow \mathcal{B}(H_{I'})$ tr-pres. cp.
s.t. $D \circ N|_{\mathcal{B}(H_{I'})} = \text{id}$ "decoding ch."

Def. $\tilde{\alpha}(S) = \max \{k : \exists \text{ Hilb sp. } H_{I'}, H_{O'}, \text{ density mat. } w \in \mathcal{B}(H_{I'} \otimes H_{O'})$

entanglement

assisted indep.

num.

channels $E_j : \mathcal{B}(H_{I'}) \rightarrow \mathcal{B}(H_I) \quad (1 \leq j \leq k)$

s.t. $((N \circ E_j) \otimes \text{id})(w)$ are mutually orthogonal }

2024-2-20 4

Def zero-error capacities

$$C_0(S) = \lim \frac{1}{n} \log \alpha(S^n)_{\text{"graph prod"}}$$

$$Q_0(S) = \lim \frac{1}{n} \log \alpha_q(S^n)$$

$$C_{0E}(S) = \lim \frac{1}{n} \log \tilde{\alpha}(S^n)$$