

Crash course on semidefinite programming

$$(M_d)_{sa} = \{X \in M_d : X = X^*\}$$

Given $C \in (M_d)_{sa}$, $\ell : (M_d)_{sa} \rightarrow \mathbb{R}^n$ (linear), $b \in \mathbb{R}^n$

suppose that we want to solve the problem:

$$(P) \begin{cases} \text{Maximize} & \text{Tr}(CX) \text{ for } X \in (M_d)_{sa} \\ \text{Subject to} & \ell(X) = b, X \geq 0 \end{cases}$$

"primal problem"

Def. a feasible solution to (P) is an X

$$\text{s.t. } \ell(X) = b \text{ \& } X \geq 0$$

a strict solution if a feas. sol. that is invertible.

Example: $\lambda_{\max}(C)$ (say $X > 0$)
maximal eigenval. of C

$$\text{Maximize } \text{Tr}(CX)$$

$$\text{Subject to } \text{Tr}(X) = 1, X \geq 0$$

We can construct another problem

take the transp. $\ell^t : \mathbb{R}^n \rightarrow (M_d)_{sa}$ (for the Tr-inner prod.)

$$(D) \begin{cases} \text{Minimize} & b^t v \text{ } (v \in \mathbb{R}^n) \\ \text{Subject to} & \ell^t(v) \geq C \end{cases}$$

"dual problem"; feasible sol. $\ell^t(v) \geq C$, strict $\ell^t(v) > C$

(Another conv. (P): $\text{Min } b^t v \dots$ (D) $\text{Max } \text{Tr}(CX)$...)

Ex. $\ell : (M_d)_{sa} \rightarrow \mathbb{R}$, $X \mapsto \text{Tr}(X) \rightsquigarrow \ell^t : r \mapsto r I_d$.

$$\begin{aligned} &\text{Minimize } r \\ &\text{Subj. to } r I_d \geq C \end{aligned} \rightsquigarrow \text{again computes } \lambda_{\max}(C)$$

Rem feasible sols X & v satisfy $\text{Tr}(CX) \leq b^t v$ (

$$\therefore b^t v = \ell(X)^t v = \text{Tr}(\ell^t(v) X) \geq \text{Tr}(CX) \\ (X \geq 0, \ell^t(v) \geq C)$$

Thm (Strong duality)

Suppose $\exists$ feasible sol to (P), strict sol. to (D)

Then $\exists X, v$ s.t. $\text{Tr}(CX) = b^t v$.
(optimal solution)

"Farkas Lemma" by Lovász

Key lem. $\ell^t(v) \geq C$ has no solution

$$\Leftrightarrow \exists X \geq 0, X \neq 0 : \ell(X) = 0, \text{Tr}(CX) \geq 0$$

Proof of Thm. Put $\text{Opt}_D = \min \{b^t v : \ell^t(v) \geq C\}$

($\tilde{D}$) " $b^t v < \text{Opt}_D, \ell^t(v) \geq C$ " is not feasible

Put $\tilde{\ell} : (M_{d+1} | s_a \rightarrow \mathbb{R}^n, Y \mapsto \ell\left(\begin{bmatrix} Y_{2,2} & \dots & Y_{3,d+1} \\ \vdots & & \vdots \\ Y_{d+1,2} & \dots & Y_{d+1,d+1} \end{bmatrix}\right) = \begin{bmatrix} b_1 Y_{11} \\ \vdots \\ b_n Y_{1n} \end{bmatrix}$

$$\tilde{C} = \begin{bmatrix} -\text{Opt}_D & 0 \\ 0 & C \end{bmatrix}$$

Then ($\tilde{D}$) is the dual prob. for $\tilde{C}, \tilde{\ell}, b$.

$$\tilde{\ell}^t(v) = \begin{bmatrix} -b^t v & 0 \\ 0 & \ell^t(v) \end{bmatrix}$$

Key lem for $\tilde{\ell}, \tilde{C}$ gives $Y \geq 0, Y \neq 0$ s.t. $\tilde{\ell}(Y) = 0$

and $\text{Tr}(\tilde{C}Y) \geq 0$

Writing $Y = \begin{bmatrix} a & 1,1^t \\ m & Z \end{bmatrix}$ $\ell(Z) = a b, \text{Tr}(CZ) \geq a \text{Opt}_D$

$a \neq 0$ by Key Lem $\Rightarrow X = a^{-1}Z$ gives an
opt. sol. for (P)

$\hookrightarrow$ otherwise $Z \geq 0, Z \neq 0, \ell(Z) = 0, \text{Tr}(CZ) = 0$ w $\ell^t(v) \geq C$ has no sol

pf of Key Lem

Step 1. $l^t(v) > 0$ has no sol $\Leftrightarrow$
 $\exists X \geq 0, X \neq 0$ s.t. $l(X) = 0$

Step 2 Claim

If $X \geq 0, X \neq 0, l(X) = 0$ & $\text{Tr}(CX) > 0$
and $l^t(v) > c$ then

$$\text{Tr}(CX) < \text{Tr}(l^t(v)X) = v^t l(X). \quad \text{contr.}$$

For the other dir. consider

$$\tilde{C} = \begin{bmatrix} -c & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{l}(Y) = \begin{bmatrix} \begin{bmatrix} Y_{11} & \dots & Y_{1d} \\ \vdots & & \vdots \\ Y_{d1} & \dots & Y_{dd} \end{bmatrix} \\ \text{Tr}(\tilde{C} Y) \end{bmatrix}$$

$$\tilde{l}^t \left(\begin{bmatrix} v \\ t \end{bmatrix} \right) = l(v) + t \tilde{C}$$

Step 1 gives $Y \geq 0, Y \neq 0$ s.t. $\tilde{l}(Y) = 0$

Writing $Y = \begin{bmatrix} X & m \\ m^t & z \end{bmatrix}$ we get

$$X \geq 0, \quad l(X) = 0, \quad \text{Tr}(CX) = z \geq 0$$

$X = 0$ would imply $z = 0 \Rightarrow Y = 0$ $\square$

