

Quantum Lovász number

Recall: entanglement assisted indep. num of channel (tr-pres. cp. map) $N: B(H_I) \rightarrow B(H_O)$

$$\chi(N) = \max \left\{ k : \exists \text{ Hilb. sp. } H_{I'}, H_{O'} \right. \\ \left. \begin{array}{l} \text{density mat. } \omega \in B(H_{I'} \otimes H_{O'}) \\ \text{channels } E_i: B(H_{I'}) \rightarrow B(H_{\pm}) \\ \text{s.t. } ((N \circ E_i) \otimes \text{id})(\omega) \in B(H_O \otimes H_{O'}) \text{ are orthog.} \end{array} \right\}$$

Prop 1 Take Stinespring dilations

$$V: H_I \rightarrow H_O^n \text{ for } N, \quad V^{(i)}: H_{I'} \rightarrow H_I^m \text{ for } E_i$$

$$\text{so } N(p) = \text{Tr}_{Mn} \left(\underbrace{V p V^*}_{\text{in } B(H_O^n) \simeq B(H_O) \otimes M_n} \right), \quad E_i(p) = \text{Tr}_{Mm} (V^{(i)} p V^{(i)*})$$

and $S = \text{span} \langle K_i^* K_j \rangle \subset B(H_I)$ is the q -graph for N if $Vx = (K_i x)_{i=1}^n$

$$\text{Then } (V^{(i)} \otimes I_{O'}) \omega (V^{(j)*} \otimes I_{O'}) \in (S \otimes B(\mathbb{C}^m) \otimes \mathbb{C} I_{O'})^\perp \text{ for } i \neq j.$$

Proof The orthogonality of $((N \circ E_i \otimes \text{id})(\omega))_i$ means $\text{Tr}_{O O'} \left(\left(\text{Tr}_n \text{Tr}_m ((V V^{(i)} \otimes I_{O'}) \omega (V^{(i)*} V^* \otimes I_{O'})) \right) \left(\text{Tr}_n \text{Tr}_m ((V V^{(j)} \otimes I_{O'}) \omega (V^{(j)*} V^* \otimes I_{O'})) \right) \right) = 0$

$$\text{LHS} = \sum_{a,b=1}^n \sum_{c,d=1}^m \text{Tr}_{O O'} \left(\langle ac | (V V^{(i)} \otimes I_{O'}) \omega (V^{(i)*} V^* \otimes I_{O'}) | ac \rangle \langle bd | \text{ same with } j | bd \rangle \right)$$

with $(|a\rangle)_{a=1}^n$ ONB of $\mathbb{C}^n$, $(|c\rangle)_{c=1}^m$ ONB of $\mathbb{C}^m$

each term is $\text{Tr}(XY)$, X, Y pos. in $B(H_O \otimes H_{O'})$

$$\Rightarrow \text{Tr}(XY) \geq 0 \text{ hence } \text{Tr}(XY) = 0,$$

i.e. $x_{ac}^{(i)} = \langle ac | (V V^{(i)} \otimes I_{O'}) \omega (V^{(i)*} V^* \otimes I_{O'}) | ac \rangle$ satisfy

$$\text{Tr}(x_{ac}^{(i)} x_{bd}^{(j)}) = 0 \quad (i \neq j)$$

writing $y_{ac}^{(i)} = \omega^{\frac{1}{2}} (V^{(i)*} V^* \otimes I_{O'}) | ac \rangle \in B(H_O \otimes H_{O'}, H_{I'} \otimes H_O)$

$$\text{so } x_{ac}^{(i)} = y_{ac}^{(i)*} y_{ac}^{(i)} \quad \hookrightarrow \omega^{\frac{1}{2}} (K_c^{(i)*} K_a^* \otimes I_{O'})$$

$$\text{we get } \text{Tr}(y_{ac}^{(i)} x_{bd}^{(j)} y_{ac}^{(i)*}) = 0 \Rightarrow y_{ac}^{(i)} y_{bd}^{(j)*} = 0$$

$$0 = \text{Tr}(y_{ac}^{(i)} y_{bd}^{(j)*}) = \text{Tr}(y_{bd}^{(j)*} y_{ac}^{(i)})$$

(cont.) the last term is

$$\begin{aligned} & \text{Tr}_{\mathcal{H} \otimes \mathcal{H}'} (\langle b d | (V V^{(j)} \otimes I_{\mathcal{H}'}) \omega (V^{(i)*} V^* \otimes I_{\mathcal{H}'}) | a c \rangle) \\ &= \text{Tr} \left((V^{(j)} \otimes I_{\mathcal{H}'}) \omega (V^{(i)*} \otimes I_{\mathcal{H}'}) \underbrace{(K_b^* K_a \otimes |c\rangle\langle d| \otimes I_{\mathcal{H}'})}_{\text{Spans } S \otimes B(\mathbb{C}^m) \otimes \mathbb{C} I_{\mathcal{H}'}} \right) \end{aligned}$$

Rem This implies $\tilde{\alpha}(N)$ only depends on S

Def. quantum Lovász number

$$\tilde{\alpha}(S) = \sup_{\substack{T \in (B(H_1) \otimes M_n)_{sa} \\ h=1,2,\dots}} \{ \| \text{id}_{H_1 \otimes \mathbb{C}^n} + T \| : T \in (S \otimes M_n)^\perp, \text{id} + T \geq 0 \}$$

Thm 1. $\tilde{\alpha}(S) \geq \tilde{\alpha}(N)$ Take $(E_i)_{i=1}^k$ for $k = \tilde{\alpha}(N)$

Proof. By Prop $V^{(i)} (\text{Tr}_{\mathcal{H}'}(\omega)) V^{(j)*} \in (S \otimes B(\mathbb{C}^m))^\perp$
 $i \neq j$

$\eta = \text{Tr}_{\mathcal{H}'}(\omega)$ is pos. $\rightarrow$ by rescaling we assume
 $\lambda_{\max}(\eta) = 1$, eigenvector v

Take $T = [V^{(i)} \eta V^{(j)*} \otimes e_{ij}]_{i \neq j} \in B(H_1^m \otimes \mathbb{C}^k) = B(H_1 \otimes \mathbb{C}^{mk})$

Step 1 $T \in (S \otimes M_{mk})^\perp$

$$\text{Tr}(T \cdot K_a^* K_b \otimes e_{cd} \otimes e_{ji}) = 0 \quad \text{if } i \neq j \quad (T_{ii} = 0)$$

when $i \neq j$, we have $\text{Tr}(V^{(i)} \text{Tr}_{\mathcal{H}'}(\omega) V^{(j)*} \underbrace{(K_a^* K_b \otimes e_{cd})}_{\in S \otimes B(\mathbb{C}^m)}) = 0$

Step 2 $\text{id}_{H \otimes \mathbb{C}^{mk}} + T \geq 0$

$$\bullet \tilde{T} = [V^{(i)} \eta V^{(i)*} \otimes e_{ii}]_{i=1}^k = x x^* \quad \text{for } x = (V^{(i)} \eta^{\frac{1}{2}} \otimes e_i)_{i=1}^k \in B(H_1', H_1 \otimes \mathbb{C}^{mk})$$

$$\bullet \text{id} + T - \tilde{T} = (V^{(i)} \eta V^{(i)*} \otimes e_{ii})_{i=1}^k \geq 0$$

block diag.

Step 3 $\| \text{id} + T \| \geq k = \tilde{\alpha}(N)$

consider $w = (V^{(i)} v \otimes e_i)_{i=1}^k$

cont. $\langle m | \text{id} + T | m \rangle \geq \langle m | \tilde{T} | m \rangle$
 $= \sum_{i,j=1}^k \langle v | \eta | v \rangle = k^2 \quad (V^{(i) *} V^{(j)} = \delta_{ij} I_{I'})$
 $\langle m | m \rangle = \sum_i \langle v | v \rangle = k. \quad \square$

Prop 2. $\Phi = \sum_i v_i \otimes v_i^* \in H_I \otimes \bar{H}_I$ (v_i ON B)
 ("maximally entangled vector")

$\Rightarrow \tilde{g}(S) = \max \{ \langle \Phi | \text{id}_{H_I} \otimes p + T' | \Phi \rangle, p \geq 0, \text{Tr } p = 1$
 $T' \in (S \otimes B(\bar{H}_I))^{\perp}, \text{id} \otimes p + T \geq 0 \}$

Proof. eigenvector v for $\lambda_{\max}(\text{id} + T)$
 is $v = (\text{id} \otimes x) \Phi \quad x \in B(\bar{H}_I, \mathbb{C}^n)$

$\|v\| = 1 \Leftrightarrow \text{Tr } x^* x = 1.$

$p = x^* x, T' = (\text{id} \otimes x^*) T (\text{id} \otimes x) \quad \square$

Thm 2. The formula from Prop 2 is the sol. of primal problem of an SDP scheme

dual problem Minimize $\|(\text{Tr}_I \otimes \text{id})(Y)\|$
 subj. to $Y \in (S \otimes B(\bar{H}_I))_{sa}, Y \geq |\Phi\rangle\langle\Phi|$

These are in strong duality ($\exists$ strict sol. for dual)

Proof. Step 1 the primal problem should look

Maximize $\text{Tr}(CX) \quad (X \in M_d)$

Subj. to $\ell(X) = b, X \geq 0.$

$C \in (M_d)_{sa}, \ell: (M_d)_{sa} \rightarrow \mathbb{R}^n$

Take $M_d = B(\bar{H}_I \oplus H_I \otimes \bar{H}_I), C = \begin{bmatrix} 0 & 0 \\ 0 & |\Phi\rangle\langle\Phi| \end{bmatrix}$

$\ell: M_d \rightarrow \mathbb{R} \oplus (S \otimes B(\bar{H}_I))_{sa}^*$ write $\ell_1(X) \oplus \ell_2(X)$

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(cont.) writing $X = \begin{bmatrix} X_{11} & * \\ * & X_{22} \end{bmatrix}$ $X_{11} \in \mathcal{B}(\bar{H}_I)$, $X_{22} \in \mathcal{B}(H_I \otimes \bar{H}_I)$

$$l_1(X) = \text{Tr}(X_{11}). \quad l_2(X) \left(\begin{smallmatrix} x \otimes y \\ \in \mathcal{S} \mathcal{B}(\bar{H}_I) \end{smallmatrix} \right) = \text{Tr}((X_{22} - \text{id} \otimes X_{11})(x \otimes y))$$

$$\text{target vec: } b = 1 \oplus 0 \in \mathbb{R} \oplus (\mathcal{S} \otimes \mathcal{B}(\bar{H}_I))^*$$

i.e. we have

$$\text{Maximize } \text{Tr}(X_{22} |\Phi\rangle\langle\Phi|) = \langle\Phi|X_{22}|\Phi\rangle$$

$$\text{Subj. to } X \geq 0, \quad \text{Tr}(X_{11}) = 1, \\ X_{22} - \text{id} \otimes X_{11} \in (\mathcal{S} \otimes \mathcal{B}(\bar{H}_I))^{\perp}$$

$$X_{11} \leftrightarrow p, \quad X_{22} = \text{id} \otimes p + T'$$

Step 2 the dual prob. is

$$\text{Minimize } b^t v \quad v \in \mathbb{R} \oplus (\mathcal{S} \otimes \mathcal{B}(\bar{H}_I))_{sa}$$

$$\text{Subj. to } l^t(v) \geq C = \begin{bmatrix} 0 & 0 \\ 0 & |\Phi\rangle\langle\Phi| \end{bmatrix}$$

$$l_1^t(r) = r \text{id}_{\bar{I}} \oplus 0, \quad l_2^t(x \otimes y) = \begin{bmatrix} -\text{Tr}(x)y & 0 \\ 0 & x \otimes y \end{bmatrix}$$

i.e. Minimize r

$$\text{Subj. to } r \text{id}_{\bar{I}} - \sum_i \text{Tr}(x_i) y_i \geq 0, \quad \sum_i x_i \otimes y_i \geq |\Phi\rangle\langle\Phi|$$

(compute $\|\sum_i \text{Tr}(x_i) y_i\|$ for $\sum x_i \otimes y_i \geq |\Phi\rangle\langle\Phi|$)

Step 3 strict sol. for dual prob. $l^t(v) > C$

Take any $z \in (\mathcal{S} \otimes \mathcal{B}(\bar{H}_I))_{sa}$ s.t. $z > C$

and any r s.t. $r > (\text{Tr}_{\bar{I}} \otimes \text{id})(z)$

$$\text{Then } l^t(r \oplus z) > C \quad \square$$