

Recap from last time

$$S \subset B(H_I)$$

$$= S^\perp \otimes M_n$$

$$\tilde{g}(S) = \sup \{ \| \text{id}_{H_I} \otimes \text{id}_n + T \| : T \in (S \otimes M_n)^\perp, \text{id} + T \geq 0 \}$$

satisfies $\tilde{g}(S) \geq \tilde{\alpha}(N)$ for N with q -graph S

$$\text{by SDP } \tilde{g}(S) = \min \{ \| (\text{Tr}_I \otimes \text{id})(Y) \| : Y \in (S \otimes B(\bar{H}_I))_{sa}, Y \geq |\Phi\rangle\langle\Phi| \}$$

$$\text{for } \Phi = \sum v_i \otimes v_i^* \in H_I \otimes \bar{H}_I, (v_i)_{i=1}^d \text{ ONB.}$$

$$\text{Today's goal: } \tilde{g}(S_1 \otimes S_2) = \tilde{g}(S_1) \tilde{g}(S_2)$$

$$\text{for } S_i \subset B(H_i) \quad i=1, 2.$$

$$\text{this would imply } \text{coE}(N) \leq \log \tilde{g}(S)$$

$$\hookrightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \log \tilde{\alpha}(N^{\otimes n}). \quad \text{zero-error capacity with entangle}$$

$$\tilde{g}(S_1 \otimes S_2) \geq \tilde{g}(S_1) \tilde{g}(S_2)$$

$$\text{take } T_i \in (S_i \otimes M_{n_i})^\perp, \text{id}_{H_i} \otimes \text{id}_{n_i} + T_i \geq 0$$

$$\text{and } \| \text{id} + T_i \| \sim \tilde{g}(S_i).$$

$$\text{Step 1 } T = (\text{id} + T_1) \otimes (\text{id} + T_2) - \text{id} \otimes \text{id}$$

$$\text{is orthog. to } S_1 \otimes M_{n_1} \otimes S_2 \otimes M_{n_2}$$

$$\text{Step 2 } \text{id} + T \geq 0$$

$$\text{Step 3 } \| \text{id} + T \| = \| \text{id} + T_1 \| \times \| \text{id} + T_2 \|.$$

$$\tilde{g}(S_1 \otimes S_2) \leq \tilde{g}(S_1) \tilde{g}(S_2)$$

$$\text{take } Y_i \in (S_i \otimes B(\bar{H}_i))_{sa}, Y_i \geq |\Phi_i\rangle\langle\Phi_i|$$

$$\text{and } \| (\text{Tr} \otimes \text{id})(Y_i) \| \sim \tilde{g}(S_i)$$

$$\text{Step 1. } |\Phi_1, \Phi_2\rangle\langle\Phi_1, \Phi_2| = |\Phi\rangle\langle\Phi| \text{ for } \Phi = \sum m_j \otimes m_j^* \\ \text{with } (m_j)_j \text{ ONB of } H_1 \otimes H_2$$

$$\text{Step 2 } Y = Y_1 \otimes Y_2 \text{ dominates } |\Phi_1\rangle\langle\Phi_1| \otimes |\Phi_2\rangle\langle\Phi_2| = |\Phi\rangle\langle\Phi|_{S_1}$$

$$\text{Step 3 } (\text{Tr}_{B(H_1)} \otimes \text{Tr}_{B(H_2)} \otimes \text{id})(Y) = (\text{Tr} \otimes \text{id})(Y_1) \otimes (\text{Tr} \otimes \text{id})(Y_2)$$

(cont.) this has norm $\|(\text{Tr} \otimes \text{id})(Y_1)\|_1 \cdot \|(\text{Tr} \otimes \text{id})(Y_2)\|_1$.

$\mathcal{S}$ corresp. to $N = \text{Tr}$

Example 1. $S = \mathcal{B}(H_I)$: we can take $Y = |I\rangle\langle I|$

$\rightarrow (\text{Tr} \otimes \text{id})(Y) = \text{id}_{H_I}$ has norm 1. $\tilde{\mathcal{S}} = 1$.

2. $S = \mathbb{C} \text{id}_{H_I}$ (corresp. to $N = \text{id}_{\mathcal{B}(H_I)}$)

take $T = [u_i^* u_j \otimes e_{ij}]_{i,j} \in M_d \otimes M_{d^2}$, $(u_i)_{i=1}^{d^2}$ ONB of M_d by unitaries

$\rightarrow \tilde{\mathcal{S}}(S) \geq \|\text{id} + T\| = d^2 = (\dim H_I)^2$

the dual problem gives opposite ineq.

3. $H_I = \mathbb{C}^2 \otimes \mathbb{C}^d$, $S = \mathbb{C} I_{2d} + \underbrace{(\mathbb{C} I_2)^\perp}_{\{X \in M_2 : \text{Tr } X = 0\}} \otimes M_d$

S contains $M_2 \otimes \mathbb{C} I_d$, dual def of $\tilde{\mathcal{S}}$ gives

$$\tilde{\mathcal{S}}(S) \leq \tilde{\mathcal{S}}(M_2 \otimes \mathbb{C} I_d) = \tilde{\mathcal{S}}(M_2) \cdot \tilde{\mathcal{S}}(\mathbb{C} I_d) = d^2$$

Duan '09 gives $\tilde{\mathcal{S}}(S) \geq d^2$ hence we have eq.

or use $S^\perp = \mathbb{C} I_2 \otimes (\mathbb{C} I_d)^\perp$ and

$T = [I_2 \otimes u_i^* u_j \otimes e_{ij}]_{i,j} \in M_2 \otimes M_d \otimes M_{d^2}$
for $(u_i)_{i=1}^{d^2} \subset M_d$