

Pushforward / pullback of quant. graphs
classically

Given $\begin{cases} \text{graph } \Gamma, \text{ vert. set } V = V_\Gamma \\ \text{map } f : V \rightarrow W \end{cases}$

$\leadsto \vec{\Gamma} = f_* \Gamma$ pushforward graph

vert. set W

edges $f(v_1) \sim_{\vec{\Gamma}} f(v_2)$ for $v_1 \sim_\Gamma v_2$

Similarly $g : U \rightarrow V$ leads to

$\overleftarrow{\Gamma} = g^* \Gamma$ pullback graph

vert. set U

edges $u_1 \sim_{\overleftarrow{\Gamma}} u_2$ whenever $g(u_1) \sim_\Gamma g(u_2)$

Analogue for quant. graphs : pullback for cp. maps

Th'm (7.5) B, C : finite C^* -algs

$\pi_0 : B \rightarrow B(H_1)$, $\sigma : C \rightarrow B(H_2)$ reps.

$S \subset B(H_2)$ quant. grph for (C, σ)

$(\sigma(C))'$ -bimod op. sys.)

$\theta : C \rightarrow B$ compl. pos. map.

$(K, V : H_1 \rightarrow K, \pi : C \rightarrow B(K))$ Stinespring
dilation for $\pi_0 \circ \theta : C \rightarrow B(H_1)$

$S_\pi \subset B(K)$ $\pi(C)'$ -bim. op. sys. row. to S

Then $\theta_* S = \overrightarrow{S}^\theta = V^* S_\pi V \subset B(H_1)$ is a $\pi_0(B)'$ -bimod.

Rem. θ need not be unital $\leadsto V$ need not be isometry.
indep. of (K, V, π)

In any case minimal model : $K = \pi(C) \vee H$

quot of $C \otimes H$ by $\langle c_1 \otimes v_1 | c_2 \otimes v_2 \rangle = \langle v_1 | \theta(c_1^* c_2) v_2 \rangle$

(! $\overrightarrow{S}^\theta$ is denoted by $\overleftarrow{S}$ in survey)

Proof. We follow Feb 13 (Th'm 6.4)

Step 1 $\hat{S}$ is a $\pi_0(B)'$ -bimod.

$$\pi_0(B)' \subset (\pi_0 \circ \theta)(C)' \xrightarrow{f} B(K) \text{ by}$$

$$p(y) \pi(x) \vee m = \pi(x) \vee y m. \quad x \in C, m \in K$$

$\leadsto p((\pi_0 \circ \theta)(C)') \subset \pi(C)'$ and the claim reduces to $\pi(C)'$ -bim prop. of S_π .

Step 2 independence on (K, V, π)

compare with the minimal model $\hat{K}, \hat{V}, \hat{\pi}$

$$u: \hat{K} \rightarrow K \text{ isomet, } u\hat{V} = V, u\hat{\pi}(x) = \pi(x)u$$

$$u^* S_\pi u = S_{\hat{\pi}} \quad \square$$

Prop. (7.8) Suppose B, C have faithful tracial funes ψ_B, ψ_C

$\theta: C \rightarrow B$ compl. pos. map

Then the map $\hat{\theta}: B \rightarrow C$ characterized by

$$\psi_B(b \theta(c)) = \psi_C(\hat{\theta}(b)c) \text{ is compl. pos.}$$

Lem $\theta: B(H_2) \rightarrow B(H_1)$ compl. pos.

with Kraus form $\theta(x) = \sum K_i^* x K_i$ $K_i: H_1 \rightarrow H_2$

Then $\hat{\theta}$ (for Tr_{H_1} & Tr_{H_2}) has Kraus

$$\text{form } \hat{\theta}(y) = \sum K_i y K_i^*$$

Def. (7.10) $\theta: C \rightarrow B$ compl. pos.

$\pi_0: B \rightarrow B(H_1)$, $\sigma: C \rightarrow B(H_2)$ reps

$S \subset B(H_1)$ quant graph for (B, π_0)

$$\leadsto \theta^* S = \hat{S}^\theta = \hat{\theta}_* S \text{ for } \hat{\theta}: B \rightarrow C$$

quant. graph for (C, σ) .

$\Downarrow$ this is denoted $\vec{S}$ in survey

Encoding maps as tr-pres cp maps.

$$\text{map } V \xrightarrow{f} W \rightsquigarrow \theta_f : C(V) \rightarrow C(W), \delta_v \mapsto \delta_{f(v)}$$

$$\Pi \text{ on } V \rightsquigarrow S_\Pi = \{ e_{v_1 v_2} : v_1 \sim_\Pi v_2 \} \subset M_V$$

$$\text{Prop. } (\theta_f)_* (S_\Pi) = S_{f_* \Pi}.$$

Proof. Stinespring dilation for θ_f is

$$\Upsilon : \ell^2(W) \rightarrow \ell^2(V \times W), \quad e_w \mapsto \sum_{\substack{v \in V \\ f(v)=w}} e_v \otimes e_w \\ (\text{or } \ell^2(\text{Graph}(f)))$$

$$\text{Direct computation: } \Upsilon^*(e_{v_1 v_2} \otimes \text{id}_W) \Upsilon = e_{f(v_1) f(v_2)} \quad \square$$

$$\text{Similarly } g : U \rightarrow V \rightsquigarrow \theta_g^* S_\Pi = S_{g^* \Pi}$$

$$\hat{\theta}_g : C(V) \rightarrow C(U) \quad \text{is} \quad \delta_v \mapsto \delta_{v \circ g} = \sum_{\substack{u \in U \\ g(u)=v}} \delta_u \\ (\text{x-hom})$$

Stinespring dil. is triv. ($K = \ell^2(U)$, $C(V) \xrightarrow{\hat{\theta}_g} B(\ell^2(U))$, ...)

$$\theta_g^* S_\Pi = (S_\Pi)_{\hat{\theta}_g} = \langle e_{u_1 u_2} : g(u_1) \sim_\Pi g(u_2) \rangle = S_{g^* \Pi}$$

Homomorphism of graphs

$$\Gamma_1 \xrightarrow{f} \Gamma_2 \quad \text{hom is given by}$$

$$\text{a map } V_{\Gamma_1} \xrightarrow{f} V_{\Gamma_2} \text{ s.t. } v \sim_{\Gamma_1} w \Rightarrow f(v) \sim_{\Gamma_2} f(w)$$

Equivalently $f_* \Gamma_1$ is a subgraph of Γ_2

This motivates: Weaver '21

$$B_i, \pi_i : B_i \rightarrow B(H_i), S_i \subset B(H_i) \quad i=1,2$$

f.d. C^* -alg, rep q -graph for (B_i, π_i)

a CP-morphism from S_1 to S_2 is a

$$\text{compl. pos. map } \theta : B_1 \rightarrow B_2 \text{ s.t. } \theta_*(S_1) \subset S_2$$

