

Bimodules for the GNS model.

Recall equivalence of following data (Feb 6 & 9)

$$(B, \psi, A)$$

$$B : \text{f.d. } C^*-alg$$

$$\psi : \text{pos. func.}$$

$$A : \text{quant. adj. mat. op.}$$

$$\hat{B}(L^2(B, \psi))$$

$$\boxed{A} \boxed{A} = A, \dots$$

$$(B, \pi, S_\pi^0)$$

$$\pi : B \rightarrow B(H) \text{ rep.}$$

$$S_\pi^0 \in B(H)$$

$$\pi(B)' S_\pi^0 \pi(B)', \pi(Q) S_\pi^0 \pi(Q)^{-1} = S_\pi^0$$

$$Q \in B$$

$$\psi(ab) = \psi(b Q a Q^{-1})$$

When π is the GNS rep.

$$H = L^2(B, \psi), \quad \pi(a) \Lambda(b) = \Lambda(ab)$$

we have a concrete desc. of S^0 :

$$S_{\pi_\psi}^0 = \{ T \in B(L^2(B, \psi)) : \boxed{A} \boxed{T} = \boxed{T} \}$$

$$\text{Key point : Prop 5.34} \quad S_{\pi_\psi} = \pi_\psi(B)' A \pi_\psi(B)'$$

$$\text{We have } \left\{ \pi_\psi(B)' = \{ \text{right mult. by } a \in B \} \right. \\ \left. \Lambda(b) \mapsto \Lambda(ba) \right.$$

$$B(L^2(B, \psi)) = \text{span of } \Lambda(a) \Lambda(b)^*$$

$$\boxed{A} \boxed{\Lambda(a) \Lambda(b)^*} = (\text{right mult by } a) \circ A \circ (\text{right mult by } b)^* \\ \in \pi(B)' A \pi(B)'$$

$$\Rightarrow \boxed{A} \boxed{T} \in \pi(B)' A \pi(B)' \quad \text{for } T \in B(L^2(B, \psi))$$

Proof of Prop 5.34 Recall the corresp. $A \mapsto S^0$

$$\mathbb{F}_{0, 1/2} : B(L^2(B, \psi)) \xrightarrow{\sim} B \otimes B^{op}, \quad \Lambda(b) \Lambda(a)^* \mapsto a^* \otimes \mathbb{F}_{1/2}(b)^{op}$$

$$A \text{ gave a proj. } e = \mathbb{F}_{0, 1/2}(A)$$

$$\pi : B \rightarrow B(H) \text{ gives } V = (\pi \otimes \pi^{op})(e) (H \otimes \bar{H})$$

(cont). up to $H \otimes \bar{H} \cong L^2(B(H), \text{Tr})$,

V corr. to $S_A \subset B(H)$ $\pi(B)'$ -bim., $\pi(Q^{-1/2}) \in S_A$
 $\pi(Q^2) S_A \pi(Q^{-2}) = S_A$

$$S_0 = \pi(Q)^{1/4} S_A \pi(Q^{1/4}) = \pi(Q^{1/2}) S_A.$$

Lem. for GNS rep. $(\pi_\psi \otimes \pi_\psi^{\text{op}})(e) \overline{L^2(B, \psi) \otimes L^2(B, \psi)}$

is $(\pi_\psi(B)' \otimes \pi_\psi(B)^{\text{op}'}) (\Lambda \otimes \Lambda)(e)$ up to

$$\overline{L^2(B, \psi)} \cong L^2(B^{\text{op}}, \psi^{\text{op}}), \quad \Lambda(b)^* \leftrightarrow \Lambda(b^{*\text{op}})$$

(unitary isom of B^{op} -modules, $a^{\text{op}} \Lambda(b)^* = \Lambda(a^* b)^*$)

Consider the "modular operators" $\nabla^z \in B(L^2(B, \psi))$

$$\nabla^2 \Lambda(a) = \Lambda(\sigma_{-i/2}(a)) = \Lambda(Q^2 a Q^{-2})$$

Lem. Up to $L^2(B(L^2(B, \psi))) \cong L^2(B, \psi) \otimes \overline{L^2(B, \psi)}$

$$\cong L^2(B, \psi) \otimes L^2(B^{\text{op}}, \psi^{\text{op}}), \quad \Lambda(b) \Lambda(a)^* \leftrightarrow \Lambda(b) \otimes \Lambda(a^{*\text{op}})$$

$\Lambda(\nabla^{-1/2} A)$ corresp. to $(\Lambda \otimes \Lambda)(e)$

$\therefore$ Writing $A = \sum \Lambda(b_i) \Lambda(a_i)^*$ we get

$$\Lambda(\nabla^{-1/2} A) \mapsto \sum_i \Lambda(\sigma_{i/2}(b_i)) \otimes \Lambda(a_i^{*\text{op}})$$

$$(\Lambda \otimes \Lambda)(e) = \sum_i \Lambda(a_i^*) \otimes \Lambda(\sigma_{-i/2}(b_i))$$

Use $e = \sum_{\text{flip}} (e) = (\sigma_2 \otimes \sigma_2)(e)$ to get eq.

(Thm 5.17)

Now we have $S_A = \pi_\psi(B)' \nabla^{-1/2} A \pi_\psi(B)'$

i.e. collection of ops $\Lambda(a) \mapsto \Lambda(Q^{-1/2} A(a b) Q^{-1/2} c)$

$\leadsto S_0 = \pi(Q^{1/2}) S_A$ is the coll. of ops

$$\Lambda(a) \mapsto \Lambda(A(a b) c)$$

this is $\pi_\psi(B)' A \pi_\psi(B)'$