

Quantum homomorphisms & isomorphisms

Fix fin. dim. C^* -algs B_i ($i=1,2$)

faithful pos. funcs $\psi_i: B_i \rightarrow \mathbb{C}$

quant. adj. mat ops $A_i \in \mathcal{B}(L^2(B_i, \psi_i))$

G_i : q-graph given by (B_i, ψ_i, A_i)

Def. a quantum homomorphism from G_1 to G_2 is given by

- C^* -alg C

- $*$ -homomorphism $\theta: B_2 \rightarrow B_1 \otimes C$

s.t. $\tilde{\theta} \in \mathcal{B}(L^2(B_2, \psi_2), L^2(B_1, \psi_1) \otimes C)$ corr to θ

satisfies $\tilde{\theta}^*(S_1' \otimes 1_C) \tilde{\theta} \leq S_2' \otimes C$

If $(b_i)_i$ is a basis of B_2 ,

$$\theta(b_i) = \sum_j b'_{ij} \otimes c_j \quad \text{for some } b'_{ij} \in B_1, c_j \in C$$

$$\text{then } \tilde{\theta} = \sum \lambda(b'_{ij}) \lambda(b_i)^* \otimes c_j \quad \text{for } \lambda(b_i)^* \lambda(b_j) = \delta_{ij}$$

$$[\text{with } S_i' = \pi_{\psi_i}(B_i) A_i \pi_{\psi_i}|_{B_i} \subset \mathcal{B}(L^2(B_i, \psi_i))]$$

Def. a quantum isomorphism from G_1 to G_2 is given by $C, \theta: B_2 \rightarrow B_1 \otimes C$

- quant. hom

$$(\psi_2 \otimes \text{id})(\theta(b)) = \psi_1(b) \otimes 1$$

$$(m^* \otimes \text{id}_C) \tilde{\theta} = (\tilde{\theta}_{13} \tilde{\theta}_{23}^*) \circ m^*$$

$$\begin{aligned} (\text{if } m^*(\lambda(b)) &= \sum_i b'_{i1} \otimes b''_{i2}, \quad (m^* \otimes \text{id}) \tilde{\theta}(\lambda(b)) \\ &= \sum \tilde{\theta}(b'_{i1})_{13} \tilde{\theta}(b''_{i2})_{23}) \end{aligned}$$

$$- \tilde{\theta}^*(S_1 \otimes C) \tilde{\theta} = S_2 \otimes C$$

Classical case : $C = \mathbb{C}$, $B_i = C(V_i)$

*-hom $\theta : C(V_2) \rightarrow C(V_1)$ corresp. to
a map $g : V_1 \rightarrow V_2$ s.t. $\theta(f)(v) = f(g(v))$

$\tilde{\theta}^* S_1 \tilde{\theta} \subset S_2$ means $v_1 \sim_{\Gamma_1} v_2 \Rightarrow g(v_1) \sim_{\Gamma_2} g(v_2)$

When θ is a quant. iso :

Suppose $\psi_2(\delta v) = p_v$. Then

$$m^* \Lambda(\delta v) = p_v^{-1} \Lambda(\delta v) \otimes \Lambda(\delta v) , \delta v^2 = \delta v$$

$$\Rightarrow |\Lambda(\theta(\delta v))| = \sum_{m \in V_1, g(m)=v} \Lambda(\delta m) \quad \text{should satisfy the same rel.}$$

$$\Rightarrow \exists! m \in V_1 \quad \text{s.t.} \quad g(m) = v , p_m = p_v$$

i.e. g gives an isom $(C(V_2), \psi_2) \cong (C(V_1), \psi_1)$

Isomorphism of quant. graphs

B_i : fin. dim. C^* -algs, $\psi_i, A_i, \dots$

consider the case $C = \mathbb{C}$

$$\leadsto \theta : B_2 \rightarrow B_1 \quad \text{* - hom}$$

Hom. cond. $\theta^* B_1 A_1 B_1 \theta \subset B_2 A_2 B_2$

$$\text{Isom. cond.} \begin{cases} - \psi_1 \circ \theta = \psi_2 \\ - m_1^* \theta = (\theta \otimes \theta) m_2^* \end{cases}$$

Prop. θ is an isom of C^* -algs, $\theta^* A_1 \theta = A_2$

Proof. Step 1. Injectivity

$\psi_2 = \psi_1 \circ \theta$ is faithful.

Step 2 Surjectivity

$\theta^*: B_1 \rightarrow B_2$ is a hom, $\psi_2 \circ \theta^* = \psi_1$
(inn. prod. with 1)

again use faithfulness

(ker θ^* is closed under $b \mapsto b^*$
 $\rightarrow$ generated by positive elems)

Step 3 $\theta^* A_1 \theta = A_2$ follows from

$$\theta^* s_1' \theta = s_2' \quad \square$$

Quantum automorphisms of finite sets.

take $B = \mathbb{C}^n$, $\psi(\delta_k) = 1$ ($k=1, \dots, n$)

and $A = \text{id}_{L^2(B)}$ "empty graph"

Prop quant. aut. of (B, ψ, A) is given by

C , $(p_{ij})_{i,j=1}^n$ s.t.

- each p_{ij} is a proj. in C

- $\sum_i p_{ij} = 1_C = \sum_j p_{ij}$ "magic square".

Proof. $\theta: \mathbb{C}^n \rightarrow \mathbb{C}^n \otimes C \rightsquigarrow$ write $\theta(\delta_j) = \sum_i \delta_i \otimes p_{ij}$

- $\delta_j^2 = \delta_j = \delta_j^*$, $\delta_i \delta_j = 0$ ($i \neq j$)

$$\Rightarrow p_{ij}^2 = p_{ij} = p_{ij}^*$$

- $\sum_j \delta_j = 1_{\mathbb{C}^n} \Rightarrow \sum_j p_{ij} = 1_C$ for each i

- $(\psi \otimes \text{id}) \theta(\delta_j) = \psi(\delta_j) 1_C = 1_C \Rightarrow \sum_i p_{ij} = 1_C$
for each j

- $m^* \theta = (\theta_{13} \theta_{23}) m^*$ also gives $(p_{ij})_i$ mut. orthog.

Example . $C = B(H)$, $p, q \in B(H)$ proj's.

$$\begin{bmatrix} \text{id}_H - p & p & 0 & 0 \\ p & \text{id}_H - p & 0 & 0 \\ 0 & 0 & \text{id}_H - q & q \\ 0 & 0 & q & \text{id}_H - q \end{bmatrix}$$

satisfies the
cond. for $n=4$.