

Hopf algebra structure behind quantum automorphisms

Last time: we saw that quant. aut.

of $(B = \mathbb{C}^n, \psi(S_i) = 1, A = \text{id}_{L^2(B, \psi)})$

is given by C^* -alg C and $\text{prjs } (p_{ij})_{i,j=1}^n \in C$

$$\text{s.t. } \sum_i p_{ij} = 1 = \sum_j p_{ij}$$

We want to consider the universal algebraic model

$\mathcal{O}(S_n^+)$: $*\text{-alg}$ with generators $(p_{ij})_{i,j=1}^n$

relations $\hookrightarrow$ allow lin comb, prod, $(ab)^* = b^* a^*$
 $a^{**} = a$

$$\text{relations: } p_{ij} = p_{ij}^* = p_{ji}^*, \quad \sum_i p_{ij} = 1 = \sum_j p_{ij}$$

and all rels from reps $\mathcal{O}(S_n^+) \rightarrow B(H)$
 (compat. with $a \mapsto a^*$)

In particular: $p_{ij} p_{kj} = 0 \quad i \neq k, \quad p_{ij} p_{ik} = 0 \quad j \neq k$

Rem • any rep $\pi: \mathcal{O}(S_n^+) \rightarrow B(H)$ gives
 a C^* -alg that gives quant. aut. of $(\mathbb{C}^n, \psi)$

• maximal commutative quotient: function algebra on symm group S_n .

Coproduct structure

$$\Delta: \mathcal{O}(S_n^+) \rightarrow \mathcal{O}(S_n^+) \otimes \mathcal{O}(S_n^+), \quad p_{ij} \mapsto \sum_{k=1}^n p_{ik} \otimes p_{kj}$$

Exercise check this is well-defined.

This describes the composition of quant. aut.

$$\pi_1: \mathcal{O}(S_n^+) \rightarrow B(H_1), \quad \pi_2: \mathcal{O}(S_n^+) \rightarrow B(H_2)$$

$$\sim (\pi_1 \otimes \pi_2) \Delta: \mathcal{O}(S_n^+) \rightarrow B(H_1 \otimes H_2)$$

corresp. to $\mathbb{C}^n \rightarrow \mathbb{C}^n \otimes \pi_2(\mathcal{O}(S_n^+)) \rightarrow \mathbb{C}^n \otimes \pi_1(\mathcal{O}(S_n^+)) \otimes \pi_2(\mathcal{O}(S_n^+))$

Antipode on $\mathcal{O}(S_n^+)$

consider the linear map $S: \mathcal{O}(S_n^+) \rightarrow \mathcal{O}(S_n^+)$
characterized by $S(p_{ij}) = p_{ji}$, $S(ab) = S(b)S(a)$

Prop. 1) $S^2 = \text{id}$

2) $m \circ (\text{id} \otimes S) \Delta$ agrees with the hom.

$$\varepsilon: \mathcal{O}(S_n^+) \rightarrow \mathbb{C}, \quad p_{ij} \mapsto \delta_{ij}$$

similarly for $m \circ (S \otimes \text{id}) \Delta$:

$$3) (\varepsilon \otimes \text{id}) \Delta = \text{id} = (\text{id} \otimes \varepsilon) \Delta.$$

i.e. ε is the counit for Δ .

Proof: 1) S^2 preserves product, $p_{ij} \mapsto p_{ij}$

2) on generators $m \circ (\text{id} \otimes S) \Delta$ acts as

$$p_{ij} \xrightarrow{\Delta} \sum_k p_{ik} \otimes p_{kj} \xrightarrow{\text{id} \otimes S} \sum_k p_{ik} \otimes p_{jk} \xrightarrow{m} \sum_k p_{ik} p_{jk}$$

$$\text{By mutual orthogonality } \sum_k p_{ik} p_{jk} = \sum_k p_{ik}^2 = \sum_k p_{ik} = 1.$$

on polynom of generators: if we know the claim for $a, b \in \mathcal{O}(S_n^+)$

$$ab \xrightarrow{\Delta} \Delta(a) \Delta(b) = a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} \xrightarrow{\text{id} \otimes S} a_{(1)} b_{(1)} \otimes S(b_{(2)}) S(a_{(2)})$$

$$\begin{aligned} \xrightarrow{m} a_{(1)} b_{(1)} S(b_{(2)}) S(a_{(2)}) &= a_{(1)} \varepsilon(b) S(a_{(2)}) \\ &= \varepsilon(b) \varepsilon(a) = \varepsilon(ab) \end{aligned}$$

3) check on generators & use similar arg as above for polynoms. $\square$