

Quantum isomorphisms / automorphisms of classical graphs.

Last time : $B = \mathbb{C}^n$, $\phi_{std} : B \rightarrow \mathbb{C}$, $\phi(\delta_i) = 1$

$\rightarrow$ the universal alg. C for quant. aut. of $(B, \phi, \text{empty graph})$ has a Hopf alg str.

Next goal : consider nontriv. graphs.

Prop 1. $\exists$ quant iso from $(\mathbb{C}^{n_1}, \phi_{std})$ to $(\mathbb{C}^{n_2}, \phi_{std})$

$$\Rightarrow n_1 = n_2$$

Proof. We can recover n from m by $mm^* = n \text{ id}$.

$\theta : B_1 \rightarrow B_2 \otimes C$ q -iso commutes with mm^*

For $\Gamma \sim_{qis} \Lambda$, we can impose $|V_\Gamma| = |V_\Lambda|$

Quantum isom. of q -graphs (corr. March 12)

a q -iso from $G_1 = (B_1, \phi_1, A_1)$ to $G_2 = (B_2, \phi_2, A_2)$

is given by $\theta : B_2 \rightarrow B_1 \otimes C$ $*$ -hom s.t.

$\tilde{\theta} \in \mathcal{B}(L^2(B_2, \phi_2), L^2(B_1, \phi_1)) \otimes C$ corr. to θ

Satisfies $\tilde{\theta}^*(S_1^0 \otimes C)\tilde{\theta} = S_2^0 \otimes C$

For $S_i^0 = \pi_{\phi_i}(B_i)' A_i \pi_{\phi_i}(B_i)' \in \mathcal{B}(L^2(B_i, \phi_i))$

(not $S_i^{0'} = \pi_{\phi_i}(B_i) A_i \pi_{\phi_i}(B_i)$)

What $\tilde{\theta}$ represents : a map $L^2(B_2) \otimes C \rightarrow L^2(B_1) \otimes C$

- right C -module hom
 - $\lambda(b) \otimes 1 \mapsto (\lambda \otimes \text{id})(\theta(b))$ } $\lambda(b) \otimes c \mapsto (\lambda \otimes \text{id})(\theta(b)(c \otimes 1))$

Prop 2. $B_1 = B_2 = \mathbb{C}^n$, $\phi_1 = \phi_2 = \phi_{std}$.

$A_1, A_2 \in \mathcal{B}(L^2(B, \phi))$ adj. mtr op. of some graphs

Γ_1, Γ_2 corresp graphs

(cont.) $\theta: B \rightarrow B \otimes C$ is a q -iso from Γ_1 to Γ_2

$$\text{iff } (A_1 \otimes 1) \tilde{\theta} = \tilde{\theta} (A_2 \otimes 1).$$

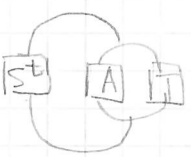
in terms of $(p_{ij})_{i,j \in C}$: $A_2 p = p A_1$.

Outline S1. $\tilde{\theta}$ is unitary $\tilde{\theta}^* \tilde{\theta} = \text{id} = \tilde{\theta} \tilde{\theta}^*$

$\tilde{\theta}^* \tilde{\theta}$ corresp. to $(p_{ij})^+ \cdot (p_{ij})$

S.2. $B(L^2(B, \phi)) \rightarrow S_i^0$, $T \mapsto \boxed{A_i} \boxed{T}$ is an orthog. proj for Tr-inn prod, A_i is theimg of $\begin{bmatrix} 1 & \dots & 1 \\ i & \dots & 1 \end{bmatrix} = \Lambda(1) \Lambda(1)^*$

$$\boxed{A_i} \boxed{T} = 0 \Leftrightarrow \forall S: \boxed{A_i} \boxed{T} \boxed{S} = 0 \Leftrightarrow \boxed{S^t} \boxed{A_i} \boxed{T} = 0$$



$$\Leftrightarrow \boxed{S^t} \boxed{A_i} \boxed{T} = 0 \Leftrightarrow \boxed{T} \boxed{A_i} \boxed{S} = 0 \Leftrightarrow T \perp S^\circ$$



S3 $T \mapsto \tilde{\theta}^* (T \otimes 1) \tilde{\theta}$ sends $\Lambda(1) \Lambda(1)^*$ to itself. 

Rem. S2 implies $S2': B(L^2(B)) \otimes C \rightarrow S_i^0 \otimes C$

$T \mapsto \boxed{A_i} \boxed{T_1} \boxed{T_2}$ is an orth proj for C -valued inner prod induced by Tr.

Rem. θ is quant iso $\Leftrightarrow \tilde{\theta}^* (S_1 \otimes 1) \tilde{\theta} \subset S_2 \otimes C$
 $\tilde{\theta} (S_2 \otimes 1) \tilde{\theta}^* \subset S_1 \otimes C$

Atserias-Mančinsk-Roberson-Šámal-Severini-Varvitsiotis '19

$\exists \Gamma, \Lambda$ with 24 vertices s.t. $\Gamma \not\cong \Lambda$ but

$\exists$ quant isom. between $(\mathbb{C}^{24}, \phi, A_\Gamma)$ and $(\mathbb{C}^{24}, \phi, A_\Lambda)$

Hopf algebraic counterpart.

Recall $\mathcal{O}(S_n^+)$ Hopf $\ast$ -alg with generators

$$p_{ij}, \text{ rel } p_{ij} = p_{ij}^2 = p_{ij}^*, \quad U = (p_{ij})_{i,j} \text{ unitary}$$

Given two graphs Γ, Λ with $|V_\Gamma| = |V_\Lambda| = n$

$\mathcal{O}(\text{Iso}^+(\Gamma, \Lambda)) =$ quotient of $\mathcal{O}(S_n^+)$ by the extra
rel. $A_\Lambda \cdot U = U A_\Gamma$ in $M_n(\mathcal{O}(\text{Iso}^+(\Gamma, \Lambda)))$.

$\leadsto$ quant. iso from Γ to $\Lambda \equiv \text{rep } \mathcal{O}(\text{Iso}^+(\Gamma, \Lambda)) \rightarrow \mathcal{B}(H)$

$$\mathcal{O}(\text{Iso}^+(\Gamma_1, \Gamma_2)) \xrightarrow{\Delta} \mathcal{O}(\text{Iso}^+(\Lambda, \Gamma_2)) \otimes \mathcal{O}(\text{Iso}^+(\Gamma_1, \Lambda))$$

$$p_{ij} \mapsto \sum_k p_{ik} \otimes p_{kj}$$

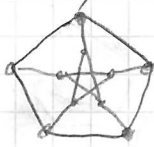
$\mathcal{O}(\text{Aut}^+(\Gamma)) = \mathcal{O}(\text{Iso}^+(\Gamma, \Gamma))$ is a Hopf $\ast$ -alg

other notations: $\mathcal{O}(\text{Qut}(\Gamma)), \mathcal{O}(\text{QAut}(\Gamma)), \dots$

Rem: When $\mathcal{O}(\text{Iso}^+(\Gamma, \Lambda)) \neq 0$, this gives a
Hopf-Galois obj. between $\mathcal{O}(\text{Aut}^+(\Gamma))$ and $\mathcal{O}(\text{Aut}^+(\Lambda))$

Rem $\mathcal{O}(\text{Aut}^+(\Gamma))$ can be noncomm or comm.
depending on Γ

- empty graph, compl. graph $\leadsto$ same as $\mathcal{O}(S_n^+)$
comm for $n \leq 3$ noncomm. for $n \geq 4$

- Schmidt '18.  gives commutative $\mathcal{O}(\text{Aut}^+(\Gamma))$
Petersen graph

- Schmidt If $\text{Aut}(\Gamma)$ has two ^{nontriv.} elems σ_1, σ_2

s.t. $\text{supp}(\sigma_1) \cap \text{supp}(\sigma_2) = \emptyset$ then

$\mathcal{O}(\text{Aut}^+(\Gamma))$ is noncomm.

$$\text{supp}(\sigma) = \{v \in V_\Gamma : \sigma(v) \neq v\}$$

