

Hilbert C^* -modules

Motivation : we have conditions like

$$\tilde{\theta}^*(S_1 \otimes C) \tilde{\theta} = S_2 \otimes C, \quad U = [p_{ij}]_{i,j} \text{ unitary}$$

for $\theta : B_2 \rightarrow B_1 \otimes C$, $p_{ij} \in C$ from quant isom.

where do we take " $\tilde{\theta}^*$ " or " U^* " ?

ad-hoc way : represent C on some H , compute adjoint in $B(H)$

systematic way : treat C , $L^2(B, \phi) \otimes C$, etc. as right C -modules with C -valued inner prod.

$\tilde{\theta}$, U are homomorphisms of these modules

$\leadsto$ adjoint is computed for the C -valued inn. prod.

Def. A : C^* -algebra (M_n , norm & $*$ -closed subalg $C(B(H), \cdot)$)

a pre-Hilbert A -module is given by :

- right A -module E

- map $E \times E \rightarrow A$ denote $\langle x, y \rangle_A$ ($x, y \in E$)

s.t. 1) conj. lin. in x , lin. in y

$$2) \langle x, y a \rangle_A = \langle x, y \rangle_A a$$

$$3) \langle y, x \rangle_A = \langle x, y \rangle_A^*$$

$$4) \langle x, x \rangle_A = a^* a \text{ for some } a$$

a Hilbert A -module is a pre-Hilb. A -mod s.t.

$\|x\| = \|\langle x, x \rangle_A\|^{1/2}$ is a Banach norm on E (completeness)

Ex. $A = M_n$, $E = \mathbb{C}^n$ as column vec sp.

$$\langle v, w \rangle_A = v^* w$$

We make $L^2(B, \phi) \otimes C$ into a Hilb. C -mod. by

$$\langle \Lambda(b_1) \otimes c_1, \Lambda(b_2) \otimes c_2 \rangle_C = \langle \Lambda(b_1) | \Lambda(b_2) \rangle C_1^* C_2$$

Given $\theta : B_2 \rightarrow B_1 \otimes C$ $*$ -hom.

$$\tilde{\theta} : L^2(B_2; \phi_2) \otimes C \rightarrow L^2(B_1; \phi_1) \otimes C, \quad \lambda(b) \otimes c \mapsto \theta(b)(1 \otimes c)$$

Prop. we have $\langle \tilde{\theta}(\lambda(b_1) \otimes c_1), \lambda(b_2) \otimes c_2 \rangle_C$
 $= \langle \lambda(b_1) \otimes c_1, \tilde{\theta}^*(\lambda(b_2) \otimes c_2) \rangle_C$

$$\theta(b) = \sum \phi(b_i^* b) b_i'' \otimes c_i \mapsto \tilde{\theta}^* = \sum |\lambda(b_i'')\rangle \langle \lambda(b_i')| \otimes c_i^*$$

Def. E_1, E_2 Hilbert A -modules, $T : E_1 \rightarrow E_2$ A -mod hom

T is adjointable if $\exists T^* : E_2 \rightarrow E_1$ s.t.

$$\langle Tx, y \rangle_A = \langle x, T^*y \rangle_A \quad x \in E_1, y \in E_2$$

T is unitary if adjointable and $TT^* = \text{id}_{E_2}$
 $T^*T = \text{id}_{E_1}$

Rem. unitary $\Leftrightarrow \langle Tx, Ty \rangle_A = \langle x, y \rangle_A \quad (x, y \in E_1)$
 and $\text{ran } T = E_2$

We will check: $\theta : B_2 \rightarrow B_1 \otimes C$ is a quant isom

from $G_1 = (B_1, \phi_1, A_1 = \text{id})$ to $G_2 = (B_2, \phi_2, A_2 = \text{id})$

$\Rightarrow \tilde{\theta}$ is unitary